



Approximations and modifications of celestial dynamics tested on the three-body system

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Abstract

Celestial Newtonian systems have regular dynamics, but with classical Keplerian rotation velocities, which disagree with the observed rotation of galaxies in the Universe. However, modifications of the classical accelerations or gravitational attractions can overcome this defect. The large-scale simulations of galaxies are with approximations with “particle–mesh” (PM) substitutions of the attractions from objects far away, which affects the regular dynamics. Here, we investigate the impact of the PM approximation and of the modifications of accelerations or gravitational attractions on the stability of the regular dynamics in a celestial system. The simple three-body system (TBS) is the simplest system to test the stability of the regular dynamics with approximations or with modifications of celestial dynamics, and it is easy to implement on a computer. Simulations of the TBS show that the PM approximation, and the modification of the accelerations (MOND), destabilizes TBS. In contrast, a modification of gravity by replacing Newton’s inverse square attraction with an increased attraction (Yukawa, MOGA) for faraway interactions stabilizes the system. The PM approximation and the MOND modification of classical dynamics do not preserve the momentum and angular momentum of a conservative system exactly, and they do not obey Newton’s third law. Although these errors and shortcomings are small, they eventually cause the instability of the regular dynamics.

Keywords Simulation of celestial systems · PM approximations · MOND · Modified gravity · Celestial regular dynamics

1 Introduction

The time evolution of astronomical systems is obtained by simulations, where the objects’ positions are calculated employing algorithms for discrete classical dynamics. This approach has been used since Newton’s time, and today, the dynamics of galaxies with billions of suns are obtained by large-scale simulations. An overview of simulations of galaxial systems is given in Springel (2016); Vogelsberger et al. (2020). The dynamic behavior of these large-scale simulations is achieved by making “particle–mesh” approximations (PM), PM3

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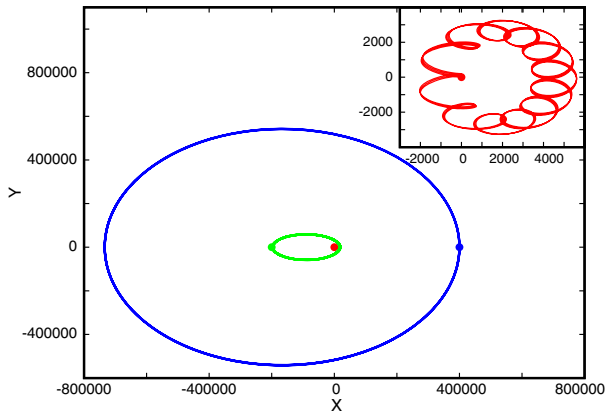


Fig. 1 Regular orbits in a classical mechanics TBS system with two light objects around a heavy object and for 10^7 discrete time steps. The ≈ 45 orbits in green are for the object No. 1 with start position at its aphelion (green dot), and in blue are the corresponding \approx four orbits for the object No. 2 with start position at its perihelion (blue dot). The red “dot” is the orbits of the heavy object No. 3 with start position at the center of mass (origin), and the inset shows the \approx four orbits of the object

(Hockney et al. 1974; Efstathiou et al. 1985) or Tree-PM (Bagla 2002), for the small effects of the many objects that are far away and which are assumed to have only a minor influence on an object’s motion. However, recent simulations indicate that these approximations used in the simulations can destabilize the systems (Toxvaerd 2025).

Large-scale simulations of galaxy models show that the simulated galaxies are unstable and have a classical Kepler-like rotation that deviates from the experimentally determined rotation velocities of galaxies. This inspired Milgrom to formulate a classical dynamics with a modified acceleration at large interstellar distances (MOND) (Milgrom 1983), and the MOND modification has later been reformulated (QuMOND) (Milgrom 2010). Another way to overcome the shortcomings in the simulated galaxies is to modify the gravitational attractions at large interstellar distances (Fischbach et al. 2001; Adelberger et al. 2003; Brandau and De Araujo 2012; Toxvaerd 2024).

Here, we investigate the impact of the particle–mesh approximations and the modifications of Newtonian dynamics on the stability of the regular dynamics in a three-body system (TBS). The investigation is for a TBS of a simple dwarf galaxy, consisting of a heavy mass center (“black hole”), and two objects in regular orbits around the center. The regular dynamics without approximations or modifications are shown in Fig. 1, and the mean distances and units are given in Table 1.

The TBS has challenged science ever since Newton formulated classical mechanics in 1687, and simultaneously solved the dynamics of two celestial bodies (Newton 1687). Newton’s solution of the dynamics of the two-body system satisfies Kepler’s three laws for the planets in the solar system. Since the planets in the Solar system move in orbits, bound rotations must be a classical mechanical solution for a system with three as well as many celestial bodies. But although one cannot solve the coupled second-order differential equations, even for just a TBS, regular solutions have been found for the system (Šuvakov and Dmitrašinović 2013, Dmitrašinović et al. 2018, Li et al. 2025). An overview of the history of the three-body system and its dynamic behavior is given in Krishnaswami and Senapati (2019), and how to simulate a TBS system is given in Appendix.

Table 1 Data for the three-body “dwarf galaxy” model

Object	Dwarf model ¹		Milky Way	
	Mean distance	Mass	Mean distance	Mass
No. 1 (“Sun”)	145000	1	83400 ²	$M_{\odot}=1$
No. 2 (Outer star)	590000	0.5		
No. 3 (“Black hole”)	3500	100	0	

Units in the TBS model: Mass in unit of the No. 1’s (“Sun”) mass $M_{\odot}=1$, length unit in pc (Toxvaerd 2022), and the gravitational constant $G = 1$

¹For set up of units in the model, see (Toxvaerd 2022)

²Distances in the Milky Way (Reid et al. 2014)

Simulations of systems with classical dynamics are performed using discrete algorithms, and almost all simulations are with Newton’s discrete algorithm (Newton 1687; Toxvaerd 2023). The classical dynamics obtained with this algorithm is time-reversible and symplectic, and has the same invariances (momentum, angular momentum, and energy) as the analytic dynamics. Hence, it is exact in the same sense as the corresponding exact solution to the coupled second-order differential equations for Newton’s analytic dynamics (Toxvaerd 2023, 2024, 2025). Here, the exact discrete dynamics for TBS are used to investigate the sensitivities of the regular orbits to the various approximations and modifications used in large-scale simulations of celestial systems’ dynamics.

2 The dynamics of a three-body system

Newtonian dynamics for celestial objects is given by Newton’s classical dynamics and his inverse square law (ISL) of gravitation

$$\mathbf{F}_i(t) = m_i \mathbf{a}_i \hat{\mathbf{a}}_i(t) = \sum_{j \neq i}^N \mathbf{f}_{ij}(t) = - \sum_{j \neq i}^N \frac{G m_i m_j}{r_{ij}^2(t)} \hat{\mathbf{r}}_{ij}(t) \tag{1}$$

for the acceleration $\mathbf{a}_i(t) = \mathbf{a}_i \hat{\mathbf{a}}_i(t)$ caused by the sum of forces $\sum_{j \neq i}^N \mathbf{f}_{ij}(t)$ on the object i in the ensemble of N objects, by baryonic objects j with mass m_j at distances $r_{ij}(t)$, and at time t .

Newton’s discrete dynamics for TBS and how to start the TBS are described in Appendix. The TBS can have solutions with regular orbits of the objects around their center of gravity. Here we want to set up a TBS with a heavy “sun” or mass center in a galaxy with two planets or stars in a galaxy, one inner and one outer object in the TBS. The TBS system with the ellipses in Fig. 1 is with (mass in units of objects’ No. 1 mass, and dynamics units given by the gravitational constant $G = 1$): $m_1 = 1, m_2=0.5$ and $m_3=100$, and with the start position of No. 1 at $[x_1(0), y_1(0)] = [-200000, 0]$, No. 2 at $[x_2(0), y_2(0)] = [400000, 0]$, and with the heavy object at the origin at the start $[x_3(0), y_3(0)] = [0, 0]$. The orbits in the figure are obtained for 10^7 time steps with $\delta t=100$ where object No. 1 in green had circled ≈ 45 times in the elliptical orbit. In blue is the corresponding ≈ 4 orbits of the light object No. 2, and the inset shows the heavy object’s more complicated four orbits in red. The TBS was simulated 10^{10} time steps, and the regular dynamics is stable.

The TBS is used for testing the stability of the system when exposed to the approximations and modifications of celestial dynamics.

3 PM approximation of faraway interactions in the three-body system

In large-scale simulations of galaxies, one sorts the N-1 forces on object i in Eq. (1) into a sum of $N'_i(t)$ short-range forces $\mathbf{f}_{ij}(r_{ij})$ with distances $r_{ij}(t)$, which are less than a certain (large) distance r_0 , and for which the forces are computed directly, and $N''_i(t)$ forces from objects far away with $r_{ij}(t) > r_0$. Their positions are interpolated onto a mesh with a certain grid length l_{grid} , and their forces are taken from the centers $\mathbf{r}_{\alpha(j)}(t)$ of the boxes $\alpha(j)$, with grid length l_{grid} and with a distance $r_{i\alpha(j)}(t)$ and direction $\hat{\mathbf{r}}_{i\alpha(j)}(t)$ to object i . The force on object j from i is correspondingly taken from the center $\mathbf{r}_{\alpha(i)}(t)$ of the subbox with object i .

The division of the space into a sum of cubes with mean field attractions to i has the consequence that

$$r_{i\alpha(j)}(t) \neq r_{\alpha(i)j}(t) \neq r_{ij}(t) \tag{2}$$

$$\hat{\mathbf{r}}_{i\alpha(j)}(t) \cdot \hat{\mathbf{r}}_{j\alpha(i)}(t) \neq -1, \tag{3}$$

and

$$\mathbf{f}_{i\alpha(j)}(t) \neq -\mathbf{f}_{j\alpha(i)}(t) \neq \mathbf{f}_{ij}(t). \tag{4}$$

Equations (2, 3) with $0 \ll r_0$ and with the size length $l_{grid} \ll r_0$ look like excellent approximations. But it has the consequence that the symmetry of pair interactions is broken and that Newton’s third law

$$\mathbf{f}_{ij}(t) = -\mathbf{f}_{ji}(t) \tag{5}$$

is no longer strictly valid for dynamics with the PM approximation of the long-range forces. The momentum and angular momentum of the system are not exactly conserved, and PM might destabilize the regular dynamics in a celestial system (Toxvaerd 2025).

3.1 Testing PM approximation on the TBS

The TBS system can be used to illustrate the destabilization of a celestial system by the PM approximation, and PM is easily implemented in TBS (see Appendix). PM primarily affects objects at distances in the outer edge of a celestial system and in the TBS system object no. 2. Figure 2 shows the distances $r_{12}(t)$, $r_{13}(t)$, and $r_{23}(t)$ for the regular orbits without PM, shown in Fig. 1. With green and red is $r_{12}(t)$, $r_{13}(t)$, respectively, and $r_{23}(t)$ is in blue. The effect of a PM approximation on the regular dynamics of the TBS system, shown in Fig. 1, is obtained for a faraway distance $r_0 = 500000$ between two objects. The faraway distance is $r_0 = 500000$, marked by a solid line in Fig. 2, and the thin solid lines are $r_0 = 500000 \pm l_{grid}$ with $l_{grid} = 10000$. This PM choice affects the faraway object, No. 2’s regular dynamics by changing the direction primarily, and a percent change in the distance on average. However, it is sufficient to disrupt the system’s stability.

Figure 3a and b illustrates the effect of the PM approximation with $r_0 = 500000$ and $l_{grid} = 10000$. Figure 3a shows the PM effect for the first 10^7 time steps. (The color of the orbits is the same as in Fig. 1.) The PM affects primarily object No. 2’s regular dynamics (in blue), and the effect on object No. 1 (in green) and the heavy object No. 3 (in red) is not visible in the figures. Object No. 2 begins to perform “revolving orbits” by being exposed to the PM, and it is released from its regular dynamics after 1.49899436×10^8 time steps (Fig. 3b).

The PM effect was tested for many other choices of r_0 and l_{grid} , and for other TBS systems. The destabilizing decreases with increasing r_0 and decreasing grid length l_{grid} . But

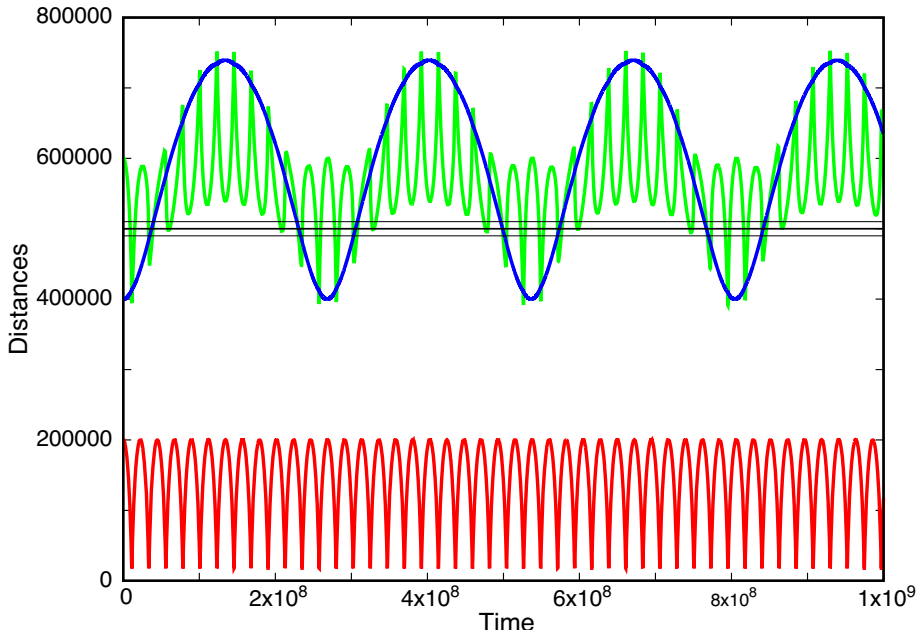


Fig. 2 Distances $r_{12}(t)$ (green), $r_{13}(t)$ (red), and $r_{23}(t)$ (blue) between the three objects as a function of time for the regular orbits, shown in Fig. 1. The thick black straight line is the borderline for the PM approximation, $r_0 = 500000$, and the thin black lines are $r_0 \pm l_{grid}$ with $l_{grid} = 10000$, used in the PM approximation

even a small change of focus for the distance r_{12} and r_{23} , and direction of the forces $\mathbf{f}_{12}(t)$ and $\mathbf{f}_{23}(t)$ are sufficient to destroy object No. 2’s regular dynamics, as illustrated in Fig. 3b.

4 Modifications of accelerations and gravitational attractions

Theoretically, one can modify the classical dynamics of a celestial system of N objects, with the acceleration \mathbf{a}_i and force \mathbf{F}_i

$$\mathbf{a}_i(t) = \mathbf{F}_i(t)/m_i \tag{6}$$

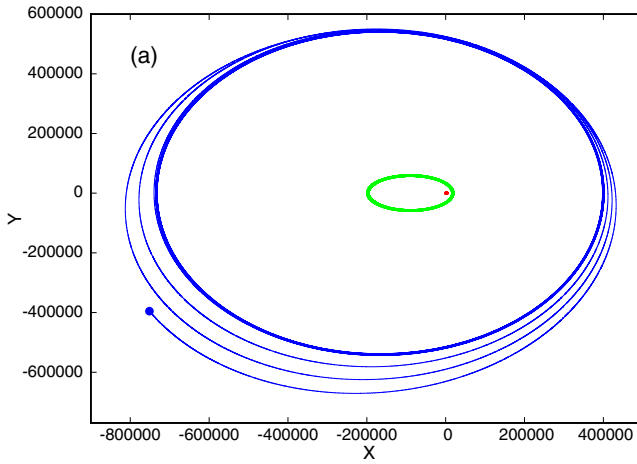
in Eq. (1) in at least two ways: by modifying the accelerations $\mathbf{a}_i(t)$: MOND, QuMOND (Milgrom 1983, 2010) or by modifying the gravitational attractions $\mathbf{F}_i(t)$: Yukawa modification (Fischbach et al. 2001) and MOGA (Toxvaerd 2024). The two kinds of modifications are qualitatively equal for a system of only two objects, but they differ significantly for a many-body system.

The modification, μ in MOND, of the acceleration $|\mathbf{a}_i|$ is obtained with the “standard interpolation function”

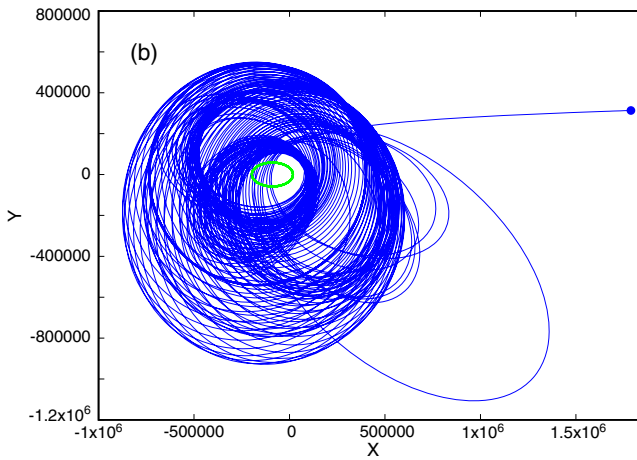
$$\mu(a/a_0) = \sqrt{\frac{1}{1 + (\frac{a_0}{a})^2}}, \tag{7}$$

or the interpolation function proposed in Gentile et al. (2011)

$$\mu(a/a_0) = \frac{|\mathbf{a}|}{|\mathbf{a}| + a_0}, \tag{8}$$



(a) 10^7 timesteps with the PM approximation.



(b) 1.49899436×10^8 timesteps with PM, and when No. 2 was released from the TBS.

Fig. 3 Dynamics of TBS with the PM approximation with $r_0 = 500000$ and $l_{grid} = 10000$. The blue orbits are for No. 2. The thick blue ellipse is without PM approximation (also shown in Fig. 1), and the orbits with thin blue are with the PM approximation. No. 1's orbits are in green with or without PM (the differences are not visible on the figures)

with the MOND modification given by

$$\mathbf{F}_i/m_i = \mathbf{a}_i \frac{|\mathbf{a}_i|}{|\mathbf{a}_i| + a_0}, \tag{9}$$

and acceleration

$$\mathbf{a}_i(\text{MOND}) = \frac{\mathbf{F}_i}{2m_i} (1 + \sqrt{1 + 4m_i a_0 / |\mathbf{F}_i|}). \tag{10}$$

MOND does not conserve momentum and angular momentum of a gravitational system because

$$\mathbf{f}_{ij}(t)\sqrt{1 + 4m_i a_0 / |F_i|} \neq -\mathbf{f}_{ji}(t)\sqrt{1 + 4m_j a_0 / |F_j|}. \tag{11}$$

The modifications, Eqs. (7) or (8), change the acceleration from the Newtonian acceleration for $|\mathbf{a}_i| \gg a_0$ at short distances to a modified acceleration

$$|\mathbf{a}_i(\text{MOND})| = \sqrt{|\mathbf{F}_i| a_0 / m_i} \tag{12}$$

for $|\mathbf{a}_i| \ll a_0$.

The asymptotic modified acceleration for an isolated object i with only one gravitational interaction, $|\mathbf{F}_i(r_{ij})| = -m_i m_j G / r_{ij}^2$, with another object, No. j , is obtained from Eqs. (1) and (10) as

$$|\mathbf{a}_i(\text{MOND})| = -\frac{\sqrt{m_j G a_0}}{r_{ij}}. \tag{13}$$

MOND is a modification of Newtonian acceleration. But in the simple case with only two objects, the modification might as well be formulated as a modification of Newton’s ISL law of universal gravitational attraction, where the inverse square attraction asymptotically is replaced with an inverse attraction (IA), Eq. (13). If this modified gravitational attraction (MOGA) is a universal law for a many-body system, the gravitational force is modified to

$$\mathbf{F}_i(\text{MOGA}) = -\sum_{j \neq i}^N \frac{m_i m_j G}{r_{ij}^2} \left(1 + \frac{r_{ij}}{r_0}\right) \hat{\mathbf{r}}_{ij}(t), \tag{14}$$

and the connection between MOND and MOGA is

$$r_0 = \sqrt{MG/a_0}, \tag{15}$$

where M is the mass which changes the acceleration.

Newtonian dynamics and Newton’s discrete algorithm (Appendix) is symplectic and with the dynamical invariances: momentum, angular momentum, and energy for a conservative system (Toxvaerd 2023). Modifications of the gravitational forces and with Newton’s discrete algorithm maintain these qualities, whereas MOND does not conserve momentum and angular momentum according to Eq. 11 (Felten 1984; Toxvaerd 2024).

The MOGA modification of Newton’s ISL to an inverse attraction for large interaction distance R is an example of the $f(R)$ modification of gravity, introduced in the standard model (Capozziello and De Laurentis 2011; Clifton et al. 2012; Capozziello et al. 2024). Simulation of large-scale celestial systems with modified gravity is described in Winther et al. (2015), and cosmological tests of modified gravity are reviewed in Koyama (2016).

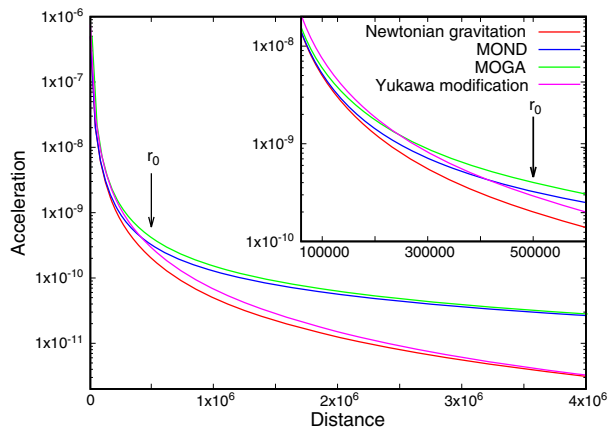
Modifications of the Newtonian ISL attraction have been proposed for a long time in an attempt to obtain the stability of galaxies by a modification of the standard model. Another modification of the ISL gravity is the “Yukawa modification” where the Newtonian gravitational ISL potential is modified by a Yukawa potential (Fischbach et al. 2001; Adelberger et al. 2003; Henrichs et al. 2021)

$$v_{ij}(r) = u(r_{ij})[1 + \alpha \exp(-r_{ij}/\lambda)], \tag{16}$$

and the corresponding modified gravitational forces are

$$\mathbf{F}_i = -\sum_{j \neq i}^N \frac{m_i m_j G}{r_{ij}^2} [1 + \alpha \exp(-r_{ij}/\lambda)(1 + r_{ij}/\lambda)] \sim -\sum_{j \neq i}^N \frac{m_i m_j G}{r_{ij}^2}. \tag{17}$$

Fig. 4 Accelerations of the outer star (No. 2 with blue in Fig. 1) as a function of the distance to object No. 3, and for the modifications of the acceleration (MOND) and the gravitational attraction (MOGA and Yukawa). Red curve is the classical Newtonian acceleration. The MOND modification of the acceleration is in blue. The MOGA modification is in green, and the Yukawa modification is in magenta



The Yukawa correction Eq. (16) modifies the attraction at medium distances and goes asymptotically over to the classical gravitational forces at large distances. The range of modification is determined by the parameters λ and modification strength α with

$$\lambda/\alpha = r_0 = \sqrt{m_j G/a_0} \tag{18}$$

The modified gravitational attraction, Eq. (16), is an example of a general modification of the force field. The parameters λ/α corresponds to the parameter r_0 in MOGA, and a possible deviation from the ISL gravity was investigated for λ in the range $\lambda \in [30, 8000]$ nm (Chen et al. 2016; Bimonte et al. 2021; Baeza-Ballesteros et al. 2022), and the rotation velocities of stars in the Milky Way were used to determine a possible Yukawa correction, Eq. (16) to the gravitational attraction (Henrichs et al. 2021). So far, however, there is no direct experimental evidence for deviation from the ISL gravitational attraction.

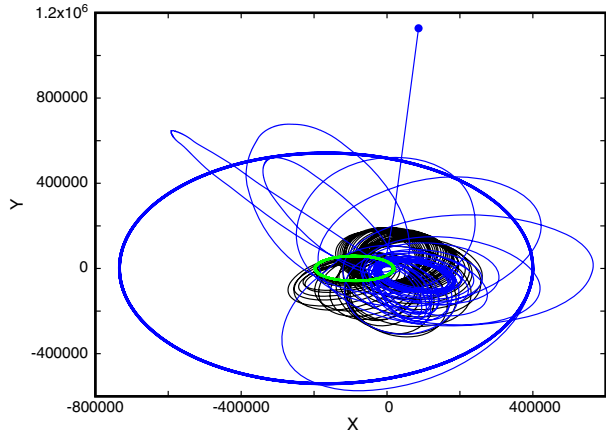
4.1 Testing MOND, Yukawa, and MOGA modifications on the TBS

The stability of the regular dynamics in TBS with modified accelerations or gravitational attractions has been studied for the MOND, Yukawa, and MOGA modifications. All three modifications increase the accelerations of objects at large distances. The different modified accelerations for the interaction with one object are shown in Fig. 4. The accelerations are for the object No. 2 with mass $m_2 = 0.5$ attracted by the heavy mass center No. 3 with a mass $m_3 = 100$ that is two hundred times heavier than the object. The acceleration is equal to the acceleration the object No. 2 obtains from the attraction of object No. 3 in the TBS.

The classical Newtonian ISL acceleration is in red in Fig. 4, and the corresponding accelerations for the various modifications with $r_0 = 500000$ are shown in blue for MOND, with magenta for the Yukawa modification Eq. (16), and in green for MOGA. The MOND and MOGA are asymptotically equal, and the classical ISL and Yukawa potential agree correspondingly asymptotically for large distances. The inset in the figure enlarges the accelerations at distances $r \leq r_0$.

Models of galaxies with classical Newtonian dynamics have a Keplerian rotation velocity of the stars, which declines with the square root of the distance to the center of rotation. This is in disagreement with experimentally determined velocities of galaxies, which are rather constant with respect to the distance to the center of rotation (Gentile et al. 2011). This motivated Milgrom to formulate the MOND modification with the increased acceleration at

Fig. 5 MOND modification for $a_0 = 4 \times 10^{-10}$ (corresponding to $r_0 = 500000$ in Yukawa, MOGA, and PM). The orbits with MOND dynamics are shown in thin blue for No. 2 and in thin black for No. 1. The position of No. 2 at $t = 1.2576 \times 10^9$ at its release is marked with a blue sphere. The ellipses with thick blue and green (from Fig. 1) are without MOND modification



large distances to overcome this shortcoming, and the enhanced accelerations of all three modifications result in a modified rotation velocity for galaxial systems in qualitative agreement with rotations of spiral galaxies in the Universe, MOND: (Gentile et al. 2011; Scherer et al. 2025); Yukawa: (Brandau and De Araujo 2012); and MOGA:(Toxvaerd 2024).

The impact of PM approximation, shown in Fig. 3a and b, is for distances $r > r_0 = 500000$, and the results reported below for the dynamics with MOND, Yukawa, and MOGA modification are correspondingly for $r_0 = 500000$, i.e., MOND: $a_0 = r_0^{-2} = 4 \times 10^{-10}$, and Yukawa: $\lambda = 500000$.

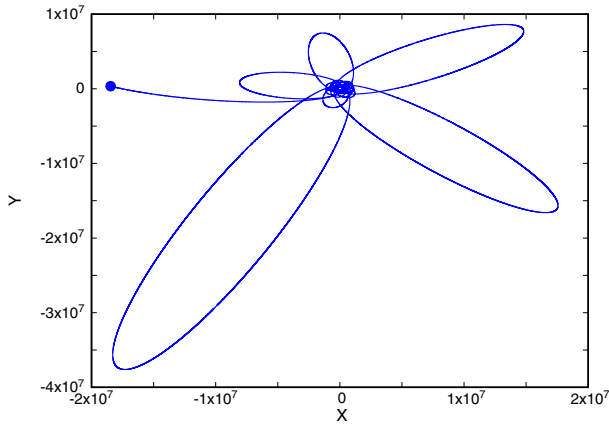
4.1.1 MOND modification of the TBS

The acceleration of one object with MOND, shown with blue in Fig. 4, is increased already for distances significantly less than $r_0 = 500000$, and the acceleration for the improved QuMOND increases the acceleration further (Scherer et al. 2025; Pflamm-Altenburg 2025; Di Cintio et al. 2025). However, the increased accelerations affect the regular dynamics of the objects in TBS and their longtime stability. Figure 5 shows the impact of MOND on the regular dynamics in the TBS. The orbits are for 1.2576×10^7 time steps with $\delta t = 100$, and at the time where object No. 2 (in blue) is released from the TBS. But also the dynamics of No. 1 nearby the center of gravity is affected by the MOND modification of the acceleration. Its orbits are shown in black in the figure.

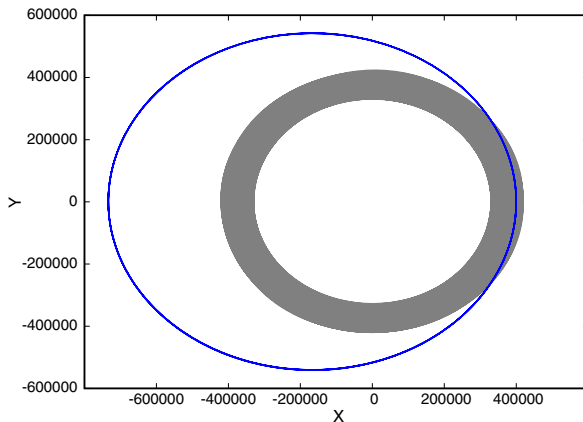
Simulations with other values of a_0 and other TBS gave the same qualitative result. The regular dynamics in TBS is sensitive to the MOND modification of the acceleration, with a destruction of the regular dynamics and the release of the objects. The destruction of the regular dynamics is due to the violation of Newton’s third law (Eq. 11), which demolishes the momentum conservation (Felten 1984; Toxvaerd 2024).

4.1.2 Yukawa modification of the TBS

The Yukawa modification is a modification of the force from a faraway object in the galaxy, and a modification of the force between pairs of objects obeys Newton’s third law and maintains all the invariances of classical mechanics, and the regular dynamics in the TBS system is, in general, preserved.



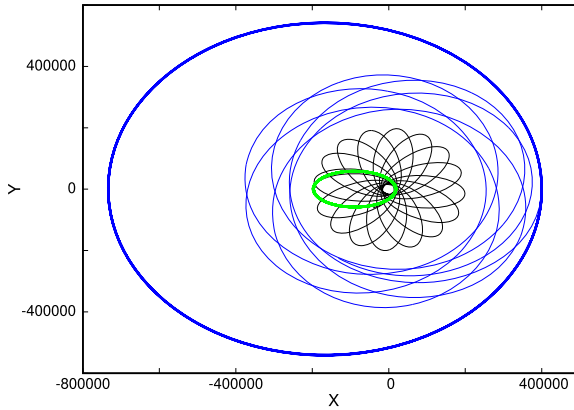
(a) The orbit of No. 2 with the Yukawa modification with $\lambda=500000$ and $\alpha = 1$. The object (blue sphere) is accelerated out of the TBS at time $t = 10^{11.1}$.



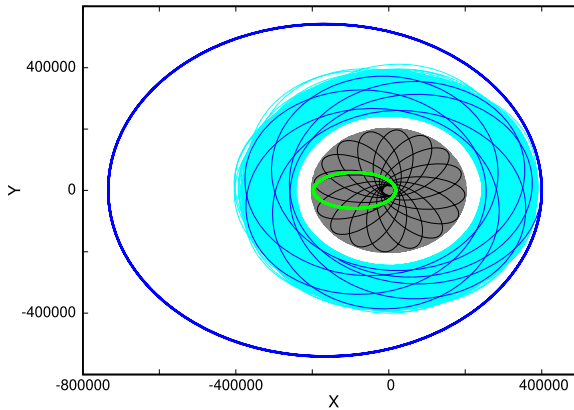
(b) The corresponding ≈ 800 stable regular revolving orbits of No. 2 with light grey for $\lambda=500000$ and $\alpha = 0.5$. The orbit in blue is the regular orbit from Figure 1 without the Yukawa modification.

Fig. 6 Simulation with the Yukawa modification for $\lambda = 500000$

The Yukawa potential Eq. 16 contains two parameters α and λ , where α is the intensity of the exponential declining modification, and λ is a measure for the range of the modification, and the net modification is given by the ratio $\lambda/\alpha = r_0$. Figure 6a and b shows the regular dynamics for two values of the intensity α of the Yukawa modification. The magenta curve in Fig. 4 is for $\alpha = 1$, and $\lambda = 500000$, and it shows the increased acceleration on the objects from the heavy center $M = m_3 = 100$ for $r \leq r_0$. But the increased accelerations can accelerate an object out of the TBS model of a dwarf galaxy, which happened for $\alpha = 1$ and with $\lambda = 500000$. The event is shown in Fig. 6a, where object No. 2 is accelerated out of the TBS system after $10^{9.1}$ time steps. However, the Yukawa accelerations result in stable regular orbits, for a reduced intensity of the Yukawa modification, which are shown in Fig. 6b for $\alpha = 0.5$ and $\lambda = 500000$.



(a) The start of MOGA. The revolving elliptical orbits are shown in black for No.1 and blue for No. 2, together with the elliptical orbits without MOGA (from Figure 1) with thick lines. The simulations are for the times where the elliptical orbits have revolved 2π .



(b) The revolving orbits with MOGA and for 10^9 timesteps. The revolving elliptical orbits with gray for No. 1 and in light blue for No. 2 are shown together with the orbits from Figure 6(a).

Fig. 7 Simulation with MOGA and for $r_0 = 500000$

The TBS system was simulated for other values of α , λ , and the Yukawa modification will, in general, stabilize the regular dynamics.

4.1.3 MOGA modification of the TBS

The corresponding impact of the MOGA modification of the attractions is shown in Fig. 7. The dynamics are for the modification distance $r_0 = 500000$, as in PM and the previous modifications. Similar to the Yukawa modification, the modification of the gravitational attractions from an inverse square attraction to an inverse attraction conserves the classical dynamics invariances, and MOGA stabilizes the regular dynamics, but with “revolving” elliptical orbits (Toxvaerd 2022). Figure 7a shows the orbits after the revolving orbits have

turned 2π around. For No.1 (in black) after $t_1 = 3.155 \times 10^6$ time steps, and after $t_2 = 2 \times t_1 = 6.310 \times 10^6$ time steps for No. 2 (in thin blue). The revolving orbits are with conservation of the length of the principal axis. Figure 7b shows the revolving orbits after 10^9 time steps, with the orbits of No. 1 in gray and light blue for No. 2.

Similar to the Yukawa modification, TBS was simulated with other value of r_0 and for other TBS systems, and with the general result that MOGA stabilizes the regular dynamics.

5 Summary of the tests and the conclusion

The purpose of this article is to investigate the effect of the PM approximations and the modifications of forces and accelerations, used in simulations of galaxies, on their regular dynamics. The three-body system (TBS) is the simplest celestial system to test the approximations and modifications used in celestial dynamics, and it is easy to implement on a computer (see Appendix).

5.1 Summary of the tests

Galaxies simulated with classical Newtonian dynamics and with PM or MOND are unstable, and the exact simulations of the TBS system show that the PM approximation, as well as the MOND modification of the accelerations, destabilizes the regular dynamics. In contrast, the Yukawa and MOGA modifications with an increased gravitational attraction stabilize the TBS and its regular dynamics. The break of momentum conservation by the PM approximations and the MOND modification, caused by a violation of Newton's third law, most likely causes the destabilization of the regular dynamics. However, all the modifications, MOND, QuMOND, Yukawa, and MOGA, increase the acceleration of faraway objects and all the modifications exhibit velocity profiles in qualitative agreement with the experimentally determined rotation velocities of galaxies.

5.2 Conclusion

Classical celestial dynamics for an N -body system have regular stable orbits. However, an extrapolation from a three-body system to a galaxy with a hundred billion stars must be taken cautiously, and only additional simulations can reveal the stability of galaxies without approximations or with modifications. But large-scale simulations without approximation are extremely time-demanding.

All of the modifications, MOND, QuMOND, Yukawa, and MOGA, increase the acceleration of distant objects, and all of the modifications exhibit velocity profiles in qualitative agreement with the experimentally determined rotation velocities of galaxies MOND: (Genile et al. 2011; Scherer et al. 2025); Yukawa: (Brandau and De Araujo 2012); and MOGA: (Toxvaerd 2024). The modifications are *ad hoc*, and only improved analyses and theories can determine whether an asymptotic modification of gravity is consistent with the current foundation of physics. In Toxvaerd (2024), we proposed that the modification of the ISL attractions is caused by a lensing and focusing by the gravitational objects in the central part of the galaxy of the gravitational waves from distant objects. A lensing that will act as a self-stabilizing effect on the halos of galaxies.

6 Appendix

6.1 Newton’s discrete algorithm

Almost all simulations of celestial systems are with Newton’s algorithm for discrete dynamics. In Newton’s discrete dynamics (Newton 1687) the time and forces are discrete with discrete force impulses at every discrete times $t, t + \delta t, t + 2\delta t, \dots$. A new position $\mathbf{r}_i(t + \delta t)$ at time $t + \delta t$ of an object i with the mass m_i is determined by the force impulse $\mathbf{f}_i(t)$ at time t acting on the object at the positions $\mathbf{r}_i(t)$, and the position $\mathbf{r}_i(t - \delta t)$ at $t - \delta t$ as

$$m_i \frac{\mathbf{r}_i(t + \delta t) - \mathbf{r}_i(t)}{\delta t} = m_i \frac{\mathbf{r}_i(t) - \mathbf{r}_i(t - \delta t)}{\delta t} + \delta t \mathbf{f}_i(t), \tag{19}$$

where the momenta $\mathbf{p}_i(t + \delta t/2) = m_i(\mathbf{r}_i(t + \delta t) - \mathbf{r}_i(t))/\delta t$ and $\mathbf{p}_i(t - \delta t/2) = m_i(\mathbf{r}_i(t) - \mathbf{r}_i(t - \delta t))/\delta t$ are constant in the time intervals in between the discrete positions.

Usually, the algorithm, Eq. (1), is presented as the leapfrog algorithm for the velocities

$$\mathbf{v}_i(t + \delta t/2) = \mathbf{v}_i(t - \delta t/2) + \delta t/m_i \mathbf{f}_i(t), \tag{20}$$

and the positions are determined from the discrete values of the momenta/velocities as

$$\mathbf{r}_i(t + \delta t) = \mathbf{r}_i(t) + \delta t \mathbf{v}_i(t + \delta t/2). \tag{21}$$

The discrete algorithm is time-reversible due to the time symmetry, which also ensures the symplecticity and the energy conservation (Toxvaerd 2023), and Newton’s third law

$$\mathbf{f}_{ij}(t) = -\mathbf{f}_{ji}(t) \tag{22}$$

ensures the conservation of momentum and angular momentum.

6.2 The three-body system

The TBS consists of three objects with masses m_1, m_2 , and m_3 . The Newtonian discrete dynamics with time increment δt is obtained from three positions $\mathbf{r}_1(t), \mathbf{r}_2(t)$, and $\mathbf{r}_3(t)$ and three velocities $\mathbf{v}_1(t - \delta t/2), \mathbf{v}_2(t - \delta t/2)$, and $\mathbf{v}_3(t - \delta t/2)$, in total six three-dimensional dynamic variables. The present TBS is started at time $t = 0$ with all three objects in their aphelion or perihelion and with start velocities in the “Ecliptica” plane, given by the plane with the three objects’ positions at the start. The three objects remain in the plane since the discrete dynamics conserves momentum, and the system is two-dimensional (2D), and with three (x,y) positions and their velocities, in total, twelve dynamic values which need to be specified at the start of the simulation. The conserved momentum and center of mass reduces the number of necessary start values to eight.

The TBS exhibits a variety of regular dynamics depending on the start values, for a review see (Krishnaswami and Senapati 2019). The present TPS system is created with two objects in elliptical orbits around a third heavy object. The objects’ six components of position $x_1(0), y_1(0), x_2(0), y_2(0), x_3(0), y_3(0)$ and six velocity components $v_{x1}(-\delta t/2), v_{y1}(-\delta t/2), v_{x2}(-\delta t/2), v_{y2}(-\delta t/2), v_{x3}(-\delta t/2), v_{y3}(-\delta t/2)$ must be specified.

Aphelion or perihelion, with the longest of the principal axes in the x-direction, determines six start values:

$$y_1(0) = y_2(0) = y_3(0) = 0, \tag{23}$$

and with

$$vx_1(0) = vx_2(0) = vx_3(0) = 0. \quad (24)$$

We need to know the velocities, not at time $t = 0$, but at time $-\delta t/2$. However, these velocities can be calculated by the algorithm and the force in the x-direction $\mathbf{f}_x(0)$ using time symmetry since

$$vx_1(\delta t/2) = -vx_1(-\delta t/2) \text{ (time reversibility),} \quad (25)$$

and (Eq. 16)

$$vx_1(\delta t/2) = vx_1(-\delta t/2) + \delta t/m_1 \mathbf{f}_{x_1}(0), \quad (26)$$

by which

$$vx_1(-\delta t/2) = -\delta t \mathbf{f}_{x_1}(0)/2m_1, \quad (27)$$

and correspondingly for the two other objects.

The conserved momentum and center of mass give two relations. Let the conserved center of mass be the origin of the 2D coordinate system, then

$$m_1x_1(t) + m_2x_2(t) + m_3x_3(t) = 0, \quad (28)$$

and the conserved momentum gives

$$m_1vy_1(t) + m_2vy_2(t) + m_3vy_3(t) = 0. \quad (29)$$

So the dynamics of the TBS system with the three objects in their aphelion or perihelion are obtained by specifying four values.

The TBS system with the ellipses in Fig. 1, used in the present investigations, is started with (mass unit given by m_1 and dynamics units given by the gravitational constant $G = 1$) with

input : $m_1, x_1(0), vy_1(-\delta t/2), m_2, m_3 = 1, -200000, 0.009, 0.5, 100,$

and with the heavy object at the origin and in rest at the start $vx_3(-\delta t/2) = vy_3(-\delta t/2) = 0$. The figures in the article are obtained by the discrete algorithm with $\delta t=100$. The result with regular dynamics shown in the figures is not sensitive to δt .

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Data Availability Data will be available on request.

Declarations

Conflict of interest The authors declare no conflict of interest.

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References

- Adelberger, E.G., Heckel, B.R., Nelson, A.E.: Tests of the gravitational inverse-square law. *Annu. Re. Nucl. Part. Sci.* **53**, 77–121 (2003)
- Baeza-Ballesteros, J., Donini, A., Nadal-Gisbert, S.: Dynamical measurements of deviations from Newton's $1/r^2$ law. *Eur. Phys. J. C* **82**, 154 (2022)
- Bagla, J.S.: TreePM: a code for cosmological N-body simulations. *J. Astrophys. Astr.* **23**, 185–196 (2002)
- Bimonte, G., Spreng, B., Maia Neto, P.A., Ingold, G.-L., Klimchitskaya, G.L., Mostepanenko, V.M., Decca, R.S.: Measurement of the Casimir force between 0.2 and 8 μm : experimental procedures and comparison with theory. *Universe* **7**, 93 (2021)
- Brandau, C.S.S., De Araujo, J.C.N.: A recipe to probe alternative theories of gravitation via N-body numerical. I. Simulations Spiral Galaxies. *Astron. J.* **750**, 29 (2012)
- Capozziello, S., De Laurentis, M.: Extended theories of gravity. *Phys. Rep.* **509**, 167–321 (2011)
- Capozziello, S., Capriolo, M., Nojiri, S.: Gravitational waves in $f(Q)$ non-metric gravity via geodesic deviation. *Phys. Lett. B* **850**, 138510 (2024)
- Chen, Y.-J., Tham, W.K., Krause, D.E., López, D., Fischback, E., Decca, R.S.: Stronger Limits on Hypotetical Yukawa Interactions in the 30–8000 nm Range. *Phys. Rev. Lett.* **116**, 221102 (2016)
- Clifton, T., Ferreira, P.G., Padilla, A., Skordis, C.: Modified gravity and cosmology. *Phys. Rep.* **513**, 1–189 (2012)
- Di Cintio, P., Re, F., Chiari, C.: Dynamical friction in the quasi-linear formulation of modified Newtonian dynamics (QuMOND). *A&A* **689**, A150 (2025)
- Dmitrašinović, V., Hudomal, A., Shibayama, M., Sugita, A.: Linear stability of periodic three-body orbits with zero angular momentum and topological dependence of Kepler's third law: a numerical test. *J. Phys. A: Math. Theor.* **51**, 315101 (2018)
- Efstathiou, G., Davis, M., Frenk, C.S., White, D.M.: Numerical techniques for large cosmological N-body simulations. *ApJS* **57**, 241–260 (1985)
- Felten, J.E.: Milgrom's revision of Newton's law: dynamical and cosmological consequences. *Astron. J.* **286**, 3–6 (1984)
- Fischbach, E., Krause, E.D., Mostepanenko, V.M., Novello, M.: New constraints on ultrashort-ranged Yukawa interactions from atomic force microscopy. *Phys. Rev. D* **64**, 075010 (2001)
- Gentile, G., Famaey, B., de Blok, W.J.: G: Things about MOND. *A&A* **527**, A76 (2011)
- Henrichs, J., Lembo, M., Iocco, F., Amendola, L.: Testing gravity with the Milky Way: Yukawa potential. *Phys. Rev. D* **104**, 043009 (2021)
- Hockney, R.W., Goel, S.P., Eastwood, J.W.: Quit high-resolution computer models of a plasma. *J. Comput. Phys.* **14**, 148–158 (1974)
- Koyama, K.: Cosmological tests of modified gravity. *Rep. Prog. Phys.* **79**, 046902 (2016)
- Krishnaswami, G., Senapati, H.: An introduction to classical three-body problem. *Resonance* **24**, 87–114 (2019)
- Li, X., Tao, Y., Li, X., Liao, S.: A numerical scheme to obtain periodic three-body orbits with finite angular momentum. *New Astron.* **119**, 102407 (2025)
- Milgrom, M.: A modification of the Newtonian dynamics: implications for galaxies. *The Astrophysical J.* **270**, 371–383 (1983)
- Milgrom, M.: Quasi-linear formulation of MOND. *MNRAS* **403**, 886–895 (2010)
- Newton, I.: *Philosophiæ naturalis principia mathematica*. *Londini, Anno MDCLXXXVII* (1687)
- Pflamm-Altenburg, J.: Numerical solutions of the complete two-body system in QUMOND. *A&A* **703**, A68 (2025)
- Reid, M.J., et al.: Trigonometric parallaxes of high mass star forming regions: The structure and kinematics of the Milky Way. *Astrophys. J.* **783**, 130 (2014)
- Scherer, D., Pflamm-Altenburg, J., Kroupa, P., Gjergo, E.: The p-Laplacian as a framework for generalizing Newtonian gravity and Milgromian gravitation. *A&A* **698**, A167 (2025)
- Springel, V.: *High Performance Computing and Numerical Modelling, Star Formation in Galaxy Evolution: Connecting Numerical Models to Reality*, p. 251. Springer, New York (2016)
- Šuvakov, M., Dmitrašinović, V.: Three classes of Newtonian three-body planar periodic orbits. *Phys. Rev. Lett.* **110**, 114301 (2013)
- Toxvaerd, S.: The stability of galaxies in an expanding Universe obtained by Newtonian dynamics. *Class. Quantum Grav.* **29**, 22500 (2022)
- Toxvaerd, S.: Planetary systems with forces other than gravitational forces. *Celest. Mech. Dyn.* **134**, 40 (2022)
- Toxvaerd, S.: Discrete molecular dynamics, comprehensive computational. *Chemistry* **3**, 329–343 (2023)
- Toxvaerd, S.: Simulations of galaxies in an expanding Universe with modified Newtonian dynamics (MOND) and with modified gravitational attractions (MOGA). *Eur. Phys. J. Plus* **139**, 395 (2024)

- Toxvaerd, S.: Energy, temperature, and heat capacity in discrete classical dynamics. *Phys. Rev. E* **109**, 015306 (2024)
- Toxvaerd, S.: Newton's algorithm for discrete classical dynamics. *J. Chem. Phys.* **162**, 024107 (2025)
- Toxvaerd, S.: Testing the approximations in large-scale simulations of systems with gravitational forces. *J. Chem. Phys.* **163**, 084101 (2025)
- Vogelsberger, M., Marinacci, F., Torrey, P., Puchwein, E.: Cosmological simulations of galaxy formation. *Nat. Rev. Phys.* **2**, 42–66 (2020)
- Winther, H.A., et al.: Modified gravity N -body code comparison project. *MNRAS* **454**, 4208–4234 (2015)

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