Five ways of deriving the equation of motion for rolling bodies

Jens Højgaard Jensen[a)]
IMFUFA and "Glass and Time," Department of Sciences, Roskilde University, DK-4000 Roskilde, Denmark

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The equation of motion for rolling bodies can be derived in several different ways. This paper shows that the equation of motion can be derived in five and only five ways without needing to know the normal and frictional forces acting at the point of contact. After enumerating these five ways, we examine two illustrative examples: an asymmetric disk rolling on a cylinder and a symmetric ball rolling on a turntable. We also discuss the educational benefits of including this topic when teaching mechanics.
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I. INTRODUCTION

In a recent paper, Turner and Turner discussed the dynamics of a rolling asymmetric body on a horizontal plane.[1] By providing a counterexample, they disproved the widespread misunderstanding that the torque equation, $d\mathbf{L}/dt = \boldsymbol{\tau}$, must always be applied about the instantaneous point of contact, see Ref. 2 for examples of the misunderstanding. Turner and Turner derived the correct equation of motion by modifying the torque equation with an additional term that they refer to as a phantom torque.

In a recent paper I discussed two equations to be used instead of $\boldsymbol{\tau} = d\mathbf{L}/dt$ in order to derive the equation of motion of rolling bodies by taking moments about the point of contact.[2] Calling the point of contact Q, the angular momentum about Q, $\mathbf{L}_Q$, apparently can be understood in two different ways leading to two different replacements of $d\mathbf{L}_Q/dt = \boldsymbol{\tau}_Q$.

If $\mathbf{L}_Q$ is understood as $\mathbf{L}_{QI} = \sum_i (\mathbf{r}_i - \mathbf{r}_Q) \times m_i \mathbf{v}_i$, the correct equation is

$$\frac{d\mathbf{L}_{QI}}{dt} = \boldsymbol{\tau}_Q + M\mathbf{v}_{CM} \times \mathbf{v}_Q. \quad (1)$$

Here, $\mathbf{r}_i$ are the position vectors in an inertial frame of reference of the mass points m_i of the rolling body, $\mathbf{r}_Q$ is the position vector of Q in the inertial frame of reference, $\mathbf{v}_i = d\mathbf{r}_i/dt$, $\mathbf{v}_Q = d\mathbf{r}_Q/dt$, M is the total mass of the body, and $\mathbf{v}_{CM}$ is the velocity of the center of mass of the body in the chosen inertial frame of reference. The index Q in $\mathbf{L}_{QI}$ refers to the point about which moments are taken. The index I indicates that the velocities in the definition of $\mathbf{L}_{QI}$ are evaluated in an inertial frame of reference.

If, on the other hand, $\mathbf{L}_Q$ is understood as $\mathbf{L}_{QQ} = \sum_i (\mathbf{r}_i - \mathbf{r}_Q) \times m_i(\mathbf{v}_i - \mathbf{v}_Q)$, the velocities thus evaluated relative to Q, the correct equation is

$$\frac{d\mathbf{L}_{QQ}}{dt} = \boldsymbol{\tau}_Q + (\mathbf{r}_{CM} - \mathbf{r}_Q) \times \left(\frac{-Md\mathbf{v}_Q}{dt}\right). \quad (2)$$

Here, as in Eq. (1), $\boldsymbol{\tau}_Q$ is the torque due to the forces existing in the inertial frame of reference. The first index Q in $\mathbf{L}_{QQ}$ again refers to the point about which moments are taken. The second index Q indicates that velocities in the definition of $\mathbf{L}_{QQ}$ are evaluated relative to Q.

In Ref. 2, it was also clarified that the contact point, Q, can be defined either as the point attached to the rolling body instantly at the point of contact or as the geometrical point defined as the point of contact at any time. What is meant by both $\mathbf{v}_Q$ and $d\mathbf{v}_Q/dt$ in Eqs. (1) and (2) depends of course on what is meant by Q. Altogether this gives four different ways of deriving the equation of motion for a rolling body without needing to know the normal and frictional forces acting at Q. Turner and Turner used Eq. (2) with Q understood as a point attached to the rolling body.

Very recently Hu has pointed out that the equation of motion of a rolling body can also be derived without needing to know the forces acting at Q by defining Q as the point attached to the ground instantly at the point of contact.[3] Differentiating $\mathbf{L}_Q$ written as a sum of its orbital part, $\mathbf{L}_{\text{orb}} = (\mathbf{r}_{CM} - \mathbf{r}_Q) \times M\mathbf{v}_{CM}$, and its spin part, $\mathbf{L}_{\text{spin}} = \mathbf{L}_{CM}$, then gives

$$\boldsymbol{\tau}_Q = (\mathbf{r}_{CM} - \mathbf{r}_Q) \times \frac{Md\mathbf{v}_{CM}}{dt} + \frac{d\mathbf{L}_{CM}}{dt}. \quad (3)$$

Thus, we have five different ways of deriving the equation of motion for a rolling body without needing to know the forces at the point of contact. The five different ways are summarized in Table I for cases with the angular momentum parallel to the angular velocity $\boldsymbol{\omega}$, as is the case, e.g., for rolling motion in two dimensions where $\mathbf{L}$ and $\boldsymbol{\omega}$ are both perpendicular to the plane of the rolling motion, independent of whether $\mathbf{L}$ is $\mathbf{L}_{QI}$, $\mathbf{L}_{QQ}$, or $\mathbf{L}_{CM}$. In Table I, $\mathbf{v}_{Qi}$ and $d\mathbf{v}_Q/dt|_i$ are the instantaneous values of $\mathbf{v}_Q$ and $d\mathbf{v}_Q/dt$, and I_Q is the moment of inertia relative to Q. For details, see Ref. 2.

Contrary to the situations where Q is defined as the point of contact at any time or a point attached to the rolling body, it makes no difference evaluating velocities relative to the laboratory or relative to Q, when Q is a point attached to the ground. Thus, we have five, not six, different ways. Since the ambiguous term "the instantaneous point of contact," Q, cannot be understood in more ways than the three mentioned and it is hard to give $\mathbf{L}_Q$ other meanings than $\mathbf{L}_{QI}$ and $\mathbf{L}_{QQ}$, there are five and only five ways of deriving the equation of motion for bodies without needing to know the forces acting at the point of contact.

Turner and Turner analyzed the motion of a semicircular hoop rocking on a horizontal plane using method **D**.[1] I demonstrated the four methods **A** through **D** used on the same case,[2] whereas Hu in his demonstration of method **E** considered the more general case of asymmetric disks rolling on an inclined plane, including the rocking hoop as a special case.[3] In Sec. II, the further generalized case of asymmetric disks rolling on cylinders is considered. The purpose is to demonstrate the value of having access to all five different ways of

Table I. Overview of the five ways of deriving the equation of motion for rolling bodies without needing to know the forces acting at the point of contact.

	Choose	Use	Note
A.	$\mathbf{L}_Q = \mathbf{L}_{QI}$; Q the point of contact at any time	Eq. (1)	$\mathbf{v}_{Qi} \neq 0$; $\mathbf{L}_{QI} = I_Q\boldsymbol{\omega}$; $\frac{dI_Q}{dt} \neq 0$ in general
B.	$\mathbf{L}_Q = \mathbf{L}_{QI}$; Q attached to the rolling body	Eq. (1)	$\mathbf{v}_{Qi} = 0$; $\mathbf{L}_{QI} = I_Q\boldsymbol{\omega} + (\mathbf{r}_{CM} - \mathbf{r}_Q) \times M\mathbf{v}_Q$; $\frac{dI_Q}{dt} = 0$
C.	$\mathbf{L}_Q = \mathbf{L}_{QQ}$; Q the point of contact at any time	Eq. (2)	$\mathbf{v}_{Qi} \neq 0$; $\mathbf{L}_{QQ} = I_Q\boldsymbol{\omega} - (\mathbf{r}_{CM} - \mathbf{r}_Q) \times M\mathbf{v}_Q$; $\frac{dI_Q}{dt} \neq 0$
D.	$\mathbf{L}_Q = \mathbf{L}_{QQ}$; Q attached to the rolling body	Eq. (2)	$\mathbf{v}_{Qi} = 0$; $\mathbf{L}_{QQ} = I_Q\boldsymbol{\omega}$; $\frac{dI_Q}{dt} = 0$; $\frac{d\mathbf{v}_Q}{dt}\vert_i \neq 0$
E.	Q attached to the ground	Eq. (3)	$\mathbf{v}_Q = 0$; $\mathbf{L}_{QI} = \mathbf{L}_{QQ}$

deriving the equation of motion in order to be able to choose the easiest one for the case in consideration. In Sec. III, a symmetric ball rolling on a turntable is discussed as a further illustration. The two cases of Secs. II and III motivate Sec. IV, where it is argued that method **A** is usually the easiest of the five ways. Section IV argues further that the seldom noticed,[2,4,5] Eq. (1) gives the easiest insight into whether the torque equation applied about the instantaneous point of contact may be used in its simple form $d\mathbf{L}_{QI}/dt = \boldsymbol{\tau}_Q$ or whether it must be supplemented with extra terms. As mentioned, the widespread justification of using the torque rule in its simple form is based on a misunderstanding. Section V discusses potential educational benefits of pointing out the misunderstanding, in university courses in mechanics, and working with its possible replacements.

The calculations in Secs. II–IV are presented in rather condensed form. The reader has to either accept the calculation details and focus on the results or take paper and pencil in order to check the calculations. The calculations are presented in condensed form in order not to overshadow the main message of the article. The main message is that working with the five different ways of deriving the equation of motion for rolling bodies illuminates how arguments in physics can be critically tested by going back to basic definitions and assumptions.

II. ROLLING OF ASYMMETRIC DISKS ON CYLINDERS

We proceed to derive the equations of motion for the asymmetric disks rolling on the cylinders shown in Fig. 1. The direction of gravity is downwards. First, we focus on the case of Fig. 1(a). If $\mathbf{x}_u$, $\mathbf{y}_u$, $\mathbf{z}_u$ are defined as unit vectors in

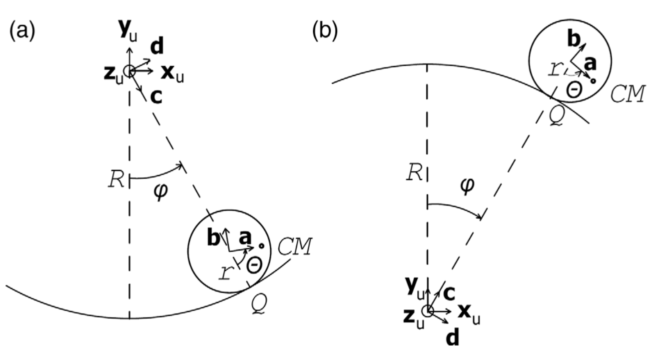

Fig. 1. (a) Asymmetric disc rolling inside a cylinder. (b) Asymmetric disc rolling on top of a cylinder. In both figures, $\mathbf{x}_u$, $\mathbf{y}_u$, and $\mathbf{z}_u$ are unit vectors in the laboratory system. Also, in both figures **a** is a unit vector directed from the centers of the disks towards the centers of mass (CM) of the disks, **b** a unit vector perpendicular to **a**, **c** a unit vector directed from the centers of the cylinders to the centers of the disks, and **d** a unit vector perpendicular to **c**. The direction of gravity is downwards.

the laboratory frame of reference as indicated in the figure, the unit vectors **a** and **c** (also indicated in the figure) are given as $\mathbf{a} = \sin(\theta+\phi)\,\mathbf{x}_u - \cos(\theta+\phi)\,\mathbf{y}_u$ and $\mathbf{c} = \sin\phi\,\mathbf{x}_u - \cos\phi\,\mathbf{y}_u$. In each of the five possible derivations, $\boldsymbol{\tau}_Q$ is consequently given by

$$\begin{aligned}\boldsymbol{\tau}_Q &= \mathbf{r}_{QCM} \times Mg(-\mathbf{y}_u) = (-r\mathbf{c} + \beta r\mathbf{a}) \times Mg(-\mathbf{y}_u)\\ &= Mgr(\sin\phi - \beta\sin(\theta+\phi))\mathbf{z}_u\,. \end{aligned}\tag{4}$$

Here, $\mathbf{r}_{QCM}$ is the vector from Q to CM (the center of mass of the disk), M is the mass of the disk, g the strength of the gravitational field, r the radius of the disk, βr is the distance of the center of mass from the geometrical center of the disk, and θ and ϕ are the angles indicated in the figure. Also the rolling condition (R being the radius of the cylinder),

$$R\frac{d\phi}{dt} = -r\frac{d\theta}{dt}, \tag{5}$$

applies for all five ways of deriving the equation of motion.

In order to continue the derivation from here one has to decide which of the five ways to follow. Here, we follow method **A**, i.e., we use Eq. (1) and identify Q as the geometrical point defined as the point of contact at any time.

Since $\mathbf{r}_{QCM}^2 = (-r\mathbf{c}+\beta r\mathbf{a})^2 = r^2(1+\beta^2-2\beta\mathbf{c}\cdot\mathbf{a}) = r^2(1+\beta^2-2\beta\cos\theta)$, we now have $dI_Q/dt = d(I_{CM} + M\mathbf{r}_{QCM}^2)/dt = 2Mr^2\beta\sin\theta\cdot d\theta/dt$. Inserting this into

$$\begin{aligned}\frac{d\mathbf{L}_{QI}}{dt} &= \frac{d\left(I_Q\left(\frac{d\theta}{dt}+\frac{d\phi}{dt}\right)\mathbf{z}_u\right)}{dt} = \frac{d\left(I_Q\left(1-\frac{r}{R}\right)\frac{d\theta}{dt}\mathbf{z}_u\right)}{dt}\\ &= \left(\frac{dI_Q}{dt}\cdot\left(1-\frac{r}{R}\right)\frac{d\theta}{dt} + I_Q\cdot\left(1-\frac{r}{R}\right)\frac{d^2\theta}{dt^2}\right)\mathbf{z}_u,\end{aligned}\tag{6}$$

where Eq. (5) was used, we get

$$\begin{aligned}\frac{d\mathbf{L}_{QI}}{dt} = \Bigg(&\left(1-\frac{r}{R}\right)\cdot I_Q\cdot\frac{d^2\theta}{dt^2}\\ &+2\left(1-\frac{r}{R}\right)\cdot\beta Mr^2\sin\theta\cdot\left(\frac{d\theta}{dt}\right)^2\Bigg)\mathbf{z}_u\end{aligned}\tag{7}$$

for the left side of Eq. (1). The third term in Eq. (1) is derived from

$$\begin{aligned}\mathbf{v}_{CM} &= \frac{d((R-r)\mathbf{c}+\beta r\mathbf{a})}{dt}\\ &= (R-r)\mathbf{d}\cdot\frac{d\phi}{dt} + \beta r\mathbf{b}\cdot\frac{d(\theta+\phi)}{dt}\\ &= \left(1-\frac{r}{R}\right)\mathbf{d}\cdot\frac{-rd\theta}{dt} + \beta r\mathbf{b}\cdot\left(1-\frac{r}{R}\right)\frac{d\theta}{dt},\end{aligned}\tag{8}$$

and

$$\mathbf{v}_Q = \frac{d(R\mathbf{c})}{dt} = R\mathbf{d} \cdot \frac{d\phi}{dt} = -r\mathbf{d} \cdot \frac{d\theta}{dt}, \tag{9}$$

where we have used Eq. (5) and $\mathbf{b}$ and $\mathbf{d}$ are unit vectors perpendicular to $\mathbf{a}$ and $\mathbf{c}$ as shown in Fig. 1(a). Since $\mathbf{b} \times \mathbf{d} = -\sin\theta \cdot \mathbf{z}_u$, we get

$$M\mathbf{v}_{CM} \times \mathbf{v}_Q = \left(1 - \frac{r}{R}\right) \cdot \beta M r^2 \sin\theta \cdot \left(\frac{d\theta}{dt}\right)^2 \cdot \mathbf{z}_u. \tag{10}$$

Inserting the results of Eqs. (4), (7), and (10) into Eq. (1), we conclude that the equation of motion for an asymmetric disk rolling inside a cylinder is

$$\left(1 - \frac{r}{R}\right) \cdot I_Q \cdot \frac{d^2\theta}{dt^2} + \left(1 - \frac{r}{R}\right) \cdot \beta M r^2 \sin\theta \cdot \left(\frac{d\theta}{dt}\right)^2 = Mgr(\sin\phi - \beta\sin(\theta + \phi)). \tag{11}$$

From this equation and the proper initial conditions, θ can be found as a function of time since $\phi = (r/R)(\theta_0 - \theta)$ according to the rolling condition in Eq. (5), where θ_0 is the value of θ when $\phi = 0$. One can also use $\theta = \theta_0 - (R/r)\phi$ to transform Eq. (11) to a differential equation involving ϕ as a function of time.

With the sign conventions of Fig. 1(b), by similar calculations, the equation of motion for an asymmetric disk rolling on top of a cylinder corresponding to Eq. (11) is found to be

$$\left(1 + \frac{r}{R}\right) \cdot I_Q \cdot \frac{d^2\theta}{dt^2} + \left(1 + \frac{r}{R}\right) \cdot \beta M r^2 \sin\theta \cdot \left(\frac{d\theta}{dt}\right)^2 = Mgr\Big(\sin(-\phi) - \beta\sin(\theta - \phi)\Big). \tag{12}$$

For $R \gg r$ where ϕ can be regarded as constant in time compared to θ and $(1 - r/R) \approx (1 + r/R) \approx 1$, both Eqs. (11) and (12) reproduce the result found by Hu[3] for an asymmetric disk rolling on an inclined plane as a limiting case.

III. SYMMETRIC BALL ROLLING ON A TURNTABLE

In order to discuss which of the five ways is the easiest it is also instructive to look at the case of a symmetric ball rolling on a turntable, as shown in Fig. 2.

We identify Q as the point of contact at any time. Denoting the radius of the ball by r, we have $\mathbf{r}_{CM} = \mathbf{r}_Q + r \cdot \mathbf{e}$, where $\mathbf{e}$ is a unit vector as shown in Fig. 2. Thus, the velocities of CM and Q relative to the laboratory $\mathbf{v}_{CM}$ and $\mathbf{v}_Q$ are equal. Since $\boldsymbol{\tau}_Q = 0$, assuming the turntable to be in a horizontal plane and $\mathbf{v}_{CM} \times \mathbf{v}_Q = 0$, Eq. (1) of method **A** simply gives $d\mathbf{L}_{QI}/dt = 0$. Denoting the mass of the ball by M and its moment of inertia around its center of mass by kMr^2, we have $\mathbf{L}_{QI} = r\mathbf{e} \times M\mathbf{v}_{CM} + kMr^2\boldsymbol{\omega}$ where $\boldsymbol{\omega}$ is the angular velocity of the ball relative to the laboratory. (The separation $\mathbf{L}_{QI} = \sum_i m_i(\mathbf{r}_i - \mathbf{r}_Q) \times \mathbf{v}_i = \sum_i m_i(\mathbf{r}_i - \mathbf{r}_{CM}) \times \mathbf{v}_i + \sum_i m_i (\mathbf{r}_{CM} - \mathbf{r}_Q) \times \mathbf{v}_i = kMr^2 \cdot \boldsymbol{\omega} + r \cdot \mathbf{e} \times M\mathbf{v}_{CM}$ of $\mathbf{L}_{QI}$ into a spin and an orbit part is valid for any chosen Q.) Inserting this equation into $d\mathbf{L}_{QI}/dt = 0$, we get

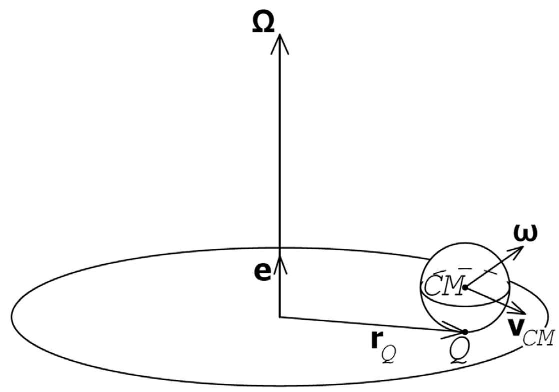

Fig. 2. Symmetric ball rolling on a turntable. Here, Ω is the angular velocity of the turntable, $\mathbf{e}$ is a unit vector in the direction of Ω, $\mathbf{r}_Q$ is the position vector of the point of contact Q, $\boldsymbol{\omega}$ is the angular velocity of the ball relative to the laboratory, and $\mathbf{v}_{CM}$ is the velocity of the center of mass (CM) of the ball relative to the laboratory.

$$r \cdot \frac{d\boldsymbol{\omega}}{dt} = -\frac{1}{k} \cdot \mathbf{e} \times \frac{d\mathbf{v}_{CM}}{dt}. \tag{13}$$

Note that this equation is nothing but a consequence of the Newtonian laws of motion (ultimately it is a consequence of Newton's second and third laws for particles since the laws for the motion of a system of particles are derived from these). In order to find the equation of motion for the ball on the turntable, besides the Newtonian constraint between $\boldsymbol{\omega}(t)$ and $\mathbf{v}_{CM}(t)$ of Eq. (13), we must add the rolling condition constraint between $\boldsymbol{\omega}(t)$ and $\mathbf{v}_{CM}(t)$. The velocity relative to the laboratory of the bottom point of the ball is $\mathbf{v}_{CM} + \boldsymbol{\omega} \times (-r\mathbf{e})$. The velocity relative to the laboratory of the point of the turntable touching the ball is $\Omega \times \mathbf{r}_Q$, where Ω is the angular velocity of the turntable. The condition for rolling without sliding is therefore

$$\Omega \times \mathbf{r}_Q = \mathbf{v}_{CM} + \omega \times (-r\mathbf{e}). \tag{14}$$

Differentiating Eq. (14) with respect to time and inserting Eq. (13), we get $\Omega \times \mathbf{v}_Q = d\mathbf{v}_{CM}/dt - (1/k) \cdot (\mathbf{e} \times d\mathbf{v}_{CM}/dt) \times (-\mathbf{e})$ and

$$\frac{d\mathbf{v}_{CM}}{dt} = \frac{k}{1+k} \cdot (\Omega \times \mathbf{v}_Q), \tag{15}$$

because $(\mathbf{e} \times d\mathbf{v}_{CM}/dt) \times \mathbf{e} = d\mathbf{v}_{CM}/dt$. Since Q was chosen to be the point of contact at any time, $\mathbf{v}_Q$ and $\mathbf{v}_{CM}$ are equal. Therefore, Eq. (15) shows that the acceleration of the CM at any time is perpendicular to the velocity of the CM and thus, of constant magnitude. Seen from the laboratory, the ball is simply performing a uniform circular motion! Equation (15) shows that the angular frequency in the circular motion is $k/(1+k) \cdot \Omega$. The center and the radius of the circle are determined by the initial conditions.

Let us denote the unknown static friction force between ball and turntable by $\mathbf{G}$. The torque rule about the center of mass gives $kMr^2 \cdot d\boldsymbol{\omega}/dt = (-r\mathbf{e}) \times \mathbf{G}$. If we combine this with Newton's second law for the motion of the center of mass, $Md\mathbf{v}_{CM} = \mathbf{G}$, Eq. (13) is arrived at. Consequently, the advantage of taking the torque about the point of contact Q and thus not needing to introduce $\mathbf{G}$ is limited in the case of a symmetric ball rolling on a turntable. My purpose of discussing this case is not to advocate for one of the five ways of taking moments about Q, but to discuss which way is the easiest among the five.

IV. THE EASIEST WAY OF DERIVING THE EQUATION OF MOTION

Which of the five ways is the easiest? Let us consider the two cases in Secs. II and III, starting with the rolling of asymmetric disks on cylinders.

As mentioned, the equations of motion (11) and (12) can be derived without needing to know the forces acting at the point of contact in four ways different from what we did above. However, the four other ways are not as easy as method **A**. Method **E** used by Hu[3] is more troublesome because calculating $d\mathbf{v}_{CM}/dt$ involves differentiation of the rotating unit vectors to second order instead of only to first order when calculating $\mathbf{v}_{CM} \times \mathbf{v}_Q$ using method **A**. According to method **D** used by Turner and Turner,[1] instead of $d\mathbf{v}_{CM}/dt$ we have to derive $d\mathbf{v}_Q/dt$. This also involves differentiation of the rotating unit vectors to second order and presents difficulties similar to those using method **E**, since Q for method **D** is understood as a point attached to the rolling body as the CM is. Method **B** leads to the same calculations as method **D**. Differentiating $\mathbf{L}_{QI} = I_Q\boldsymbol{\omega} + (\mathbf{r}_{CM} - \mathbf{r}_Q) \times M\mathbf{v}_Q$ with respect to time and inserting the result into Eq. (1) results in the same equation as a starting point for further calculations as inserting $\mathbf{L}_{QQ} = I_Q\boldsymbol{\omega}$ into Eq. (2). Finally, method **C** leads to the same calculations as method **A**. Differentiating $\mathbf{L}_{QQ} = I_Q\boldsymbol{\omega} - (\mathbf{r}_{CM} - \mathbf{r}_Q) \times M\mathbf{v}_Q$ with respect to time and inserting the result into Eq. (2) gives the same equation as a starting point for further calculations as inserting $\mathbf{L}_{QI} = I_Q\boldsymbol{\omega}$ into Eq. (1).

The five different ways of deriving the equation of motion for rolling bodies without needing to know the normal and frictional forces acting at the point of contact thus reduce to three different sets of calculations. In the case of rolling of asymmetric disks on cylinders the three sets of calculations differ quite a bit. Using method **E** and method **D** (or method **B** including the same calculations) involves differentiation of the rotating unit vectors to second order, whereas method **A** (or method **C** including the same calculations) involves only differentiation to first order. As method **C** is conceptually more awkward than method **A**, this makes method **A** the easiest way of the five for the case of rolling of asymmetric disks on cylinders.

We now turn to the symmetric ball rolling on a turntable. Besides following method **A** as in Sec. III, it is also possible to use method **C**. Here Q is also understood as the point of contact at any time and therefore $\mathbf{r}_{CM} = \mathbf{r}_Q + r\mathbf{e}$ and $\mathbf{v}_{CM} = \mathbf{v}_Q$. Consequently, we have $L_{QQ} = kMr^2\boldsymbol{\omega}$ (when velocities are evaluated relative to Q and Q follows the CM, the orbital part of the angular momentum is zero). Inserting this into Eq. (2) with $\boldsymbol{\tau}_Q = 0$ and $\mathbf{v}_Q = \mathbf{v}_{CM}$ we get again Eq. (13). But if Q is understood as a point attached to the ball or as a point attached to the turntable it is only instantaneously that $\mathbf{r}_{CM}$ equals $\mathbf{r}_Q + r\mathbf{e}$, and $\mathbf{v}_{CM}$ and $\mathbf{v}_Q$ are therefore generally not identical. For this reason, in the case of a symmetric ball rolling on a turntable, methods **B**, **D**, and **E** are difficult to apply.

We could also ask which is the easiest way of justifying the use of $d\mathbf{L}_Q/dt = \boldsymbol{\tau}_Q$ for the case of rolling of symmetric disks on an inclined plane. This is the special case of rolling of asymmetric disks on cylinders defined by $\beta = 0$ and $r/R = 0$ and the textbook case where the application of the normal torque rule about the instantaneous point of contact (Q) has often been justified by the incorrect argument that Q can be considered a fixed point since $\mathbf{v}_{Qi} = 0$. From Eq. (11), putting $\beta = 0$ and $r/R = 0$ it is seen that $d\mathbf{L}_Q/dt = \boldsymbol{\tau}_Q$ leads to the correct equation of motion,

$$I_Q \cdot \frac{d^2\theta}{dt^2} = Mgr \cdot \sin\theta. \tag{16}$$

Using the simple torque rule leads to the correct equation in this case. But how do we give a correct justification for doing it? Since $\mathbf{L}_Q$ can be understood in two ways and Q in three ways, each of the five ways mentioned may offer the justification. Let us restrict ourselves to the three methods **A**, **D**, and **E**. Methods **B** and **C** are more complicated and do not bring new perspectives. Why does the simple torque rule give the correct result? According to method **A** and Eq. (1), it is because $\mathbf{v}_{CM} \times \mathbf{v}_Q$ in this case is zero since $\mathbf{v}_{CM}$ and $\mathbf{v}_Q$ are parallel, Q being the geometrically defined point of contact. According to method **D** and Eq. (2) it is because $(\mathbf{r}_{CM} - \mathbf{r}_Q) \times (-Md\mathbf{v}_Q/dt)$ is zero since $\mathbf{r}_{CM} - \mathbf{r}_Q$ and $Md\mathbf{v}_Q/dt|_i$ are parallel, Q being now attached to the rolling body. According to method **E** it is because $|(\mathbf{r}_{CM} - \mathbf{r}_Q) \times Md\mathbf{v}_{CM}/dt| = Mr^2 \cdot d^2\theta/dt^2$, using the rolling condition $v_{CM} = r \cdot d\theta/dt$, and $|d\mathbf{L}_{CM}/dt| = I_{CM} \cdot d^2\theta/dt^2$, which combined with $I_Q = Mr^2 + I_{CM}$ gives Eq. (16) from Eq. (3). Clearly the three ways are quite different, with **A** being, perhaps, the easiest of them.

The three cases investigated thus indicate method **A** to be the easiest way. It is interesting that method **A** involves the seldom noticed Eq. (1). More generally, this equation also gives the easiest insight into whether the torque rule applied about the instantaneous point of contact may be used in its simple form $d\mathbf{L}_{QI}/dt = \boldsymbol{\tau}_Q$ or whether extra terms must be added. Deviations from the simple torque rule arise from one of two causes. Either $\mathbf{v}_{CM}$ and $\mathbf{v}_Q$ are not parallel and the term $M\mathbf{v}_{CM} \times \mathbf{v}_Q$ in Eq. (1) is consequently not zero because the rolling body is asymmetric, or $M\mathbf{v}_{CM} \times \mathbf{v}_Q$ is nonzero because the body rolls on a curved surface giving rise to a component of $\mathbf{v}_{CM}$ perpendicular to $\mathbf{v}_Q$. This applies even for symmetric bodies.

Awareness of the different possibilities of deriving the equation of motion for rolling bodies without needing to know the forces acting at the point of contact could perhaps in some cases be of greater importance than in the cases treated here, e.g., when dealing with a more complicated rolling in three dimensions. It is therefore important to note that Eqs. (1)–(3) apply for rolling in three dimensions, as well as for rolling in two dimensions.

V. POTENTIAL EDUCATIONAL YIELDS: ILLUSTRATING PHYSICS AS A RATIONALISTIC SCIENCE

The main message of the article is not the calculations themselves but the potential educational yields of dealing with the calculations and their results. Working with rolling problems and investigating the relative applicability of the five methods will obviously give students extensive mathematical training. But besides this it also gives the students insights into the nature of physics as a scientific discipline.

In my own teaching at the third year at Roskilde University, I show the students how to derive Eq. (15) and thus predict that the ball on the turntable performs uniform circular motion and does not fly off the edge. I then emphasize to the students that this surprising result is ultimately deduced from nothing but Newton's second law for the motion of point particles and Newton's third law for the action and reaction between two point particles (assuming that the static friction coefficient is high enough to insure rolling without sliding). Thus, going from Newton's second and third laws for point particles all the way to the uniform circular motion on the turntable is merely a matter of mathematical deductions.

Physics is often characterized as an empirical science. The case of rolling bodies demonstrates that physics is also a rationalistic science. In common with chemistry and biology, physics has an empirical aspect. Thereby, physics leans on the philosophical doctrine of empiricism (that all knowledge of matters of facts derives from experience and mind is not furnished with a set of concepts in advance of experience). But physics also makes common cause with mathematics by being a rationalistic science, leaning on the philosophical doctrine of rationalism (that the exercise of reason, rather than experience, provides the primary basis for knowledge). These two aspects of physics, the empirical and the rationalistic, are equally important and must both be included to fully characterize the scientific nature of physics.

Not only physics but also physicists like myself may act extremely rationalistically. When working on writing up the motion of the ball rolling on the turntable, I first made the calculations in the rotating frame of reference. These were troublesome calculations involving the integration of the different torque contributions from the centrifugal and Coriolis forces acting on the different parts of the ball. Eventually, I found the track on the turntable to be a cycloid. From this I deduced the surprising circular motion seen from the laboratory frame of reference. Then—and only then—did I place a ping-pong ball on my rotating gramophone disc and confirm the prediction. Assisted by a colleague I then performed the much easier calculations in the laboratory system. I also found that my "discovery" was in fact a rediscovery of a rediscovery. The circular orbits had been reported by Weltner[6] in 1979 followed by a note by Romer[7] in 1981 pointing out that the phenomenon dates back in the literature to at least 1844. In the note Romer also mentioned that even for him it was only late in his work with the phenomenon that he tested the theoretical prediction of circular orbits experimentally.

The point of the story about the ball on the turntable is the emphasis on the possibility of demonstrating the rationalistic side of physics when teaching classical mechanics. The subject of this article is also suitable for this purpose. For undergraduate students, a presentation of the five derivation methods described here can be used as an example to demonstrate that textbooks—even in physics—are not always correct. It can also be used to demonstrate that in a subject like classical mechanics, mistakes can be straightened out by critical analysis, building on the basic definitions and assumptions. For graduate students in physics, rolling bodies give an example of experiencing physics as a subject where arguments can be critically tested. By working with rolling problems of all kinds, investigating the internal consistency among the five methods, students by themselves can experience physics as a rationalistic science.

VI. SUMMARY

The equation of motion for rolling bodies can be derived in five different ways without needing to know the normal and frictional forces acting at the point of contact. In general, it is not correct to just equate the rate of change of the angular momentum about the point of contact with the resulting moments about the point of contact of the forces operating in the laboratory frame of reference. The five ways differ because both the term "the point of contact" and the term "angular momentum" are ambiguous. The point of contact Q can be understood as a point attached to the rolling body, as a point attached to the ground, or as the geometrically defined point of contact at any time. Moreover, $\mathbf{L}_Q$ can be understood as both $\mathbf{L}_{QI} = \sum_i (\mathbf{r}_i - \mathbf{r}_Q) \times m_i \mathbf{v_i}$ and $\mathbf{L}_{QQ} = \sum_i (\mathbf{r}_i) - \mathbf{r}_Q \times m_i(\mathbf{v}_i - \mathbf{v}_Q)$.

The easiest way of the five ways is usually to choose Q as the geometrically defined point of contact at any time and identify $\mathbf{L}_{QI}$ with $\mathbf{L}_Q$. This implies the use of the seldom-noticed Eq. (1).

The five different ways of deriving the equation of motion for rolling bodies can be used as a case for graduate students to work with for themselves and for undergraduate students to be told about, in both cases with the purpose of illuminating how arguments in physics can be critically tested by going back to the basic definitions and assumptions.

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[a)]Electronic mail: jhj@ruc.dk

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