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Hopping Models and ac Universality

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Some general relations for hopping models are established. We proceed to discuss the universality of the ac conductivity which arises in the extreme disorder limit of the random barrier model. It is shown that the relevant dimension entering into the diffusion cluster approximation (DCA) is the harmonic (fracton) dimension of the diffusion cluster. The temperature scaling of the dimensionless frequency entering into the DCA is discussed. Finally, some open problems regarding ac universality are listed.

Introduction Hopping models have been studied for many years in different contexts [1–4]. Our main concern here is the modeling of ac conduction in disordered solids, a field where hopping models are quite successful. There is a wealth of ac data for various disordered solids like ionic conductive glasses, amorphous semiconductors, polymers, etc. These solids behave similarly; from bulk ac data it is not possible to distinguish ionic conduction from electronic conduction. The quasi-universal ac features are [5]: At low frequencies the conductivity is constant, at high frequency it follows an approximate power law with an exponent which is less than one but slowly approaches one as frequency increases. As temperature is lowered the dc conductivity goes to zero and the frequency exponent goes to one (at any fixed frequency much below phonon frequencies). The frequency marking onset of ac conduction is proportional to the dc conductivity; moreover the dc conductivity is strongly temperature dependent (usually Arrhenius).

In this paper we first establish hopping models as part of a more general framework. We then proceed to discuss the extreme disorder limit of hopping models. In this limit the ac conductivity in properly scaled units becomes independent of the jump frequency probability distribution. Finally, new results are established for the diffusion cluster approximation, a recently proposed analytical approximation for the universal ac conductivity.

Diffusion and Hopping Models First, we briefly consider macroscopic diffusion in general and define the wave-number and frequency-dependent diffusion constant. Then we specialize to hopping of non-interacting particles on a cubic lattice and derive a number of relations. Many of the points made below can be found in the literature, but to our knowledge they have never been concisely written down aimed at hopping models.

If ρ is the particle density and $\mathbf{J}$ the particle current density, the most general diffusion equation obeying linearity, causality, and space-time homogeneity is

$$\mathbf{J}(\mathbf{r}, t) = -\int \mathrm{d}\mathbf{r}' \int_0^\infty \mathrm{d}t'\; D(\mathbf{r}', t')\, \nabla\rho(\mathbf{r}-\mathbf{r}', t-t')\,. \tag{1}$$

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It is convenient to introduce the so-called Laplace frequency s given by $s = i\omega$ where ω is the ordinary frequency. The wave-number and frequency-dependent diffusion constant, $D(\mathbf{k}, s)$, is by definition the Fourier-Laplace transform of $D(\mathbf{r}', t')$

$$D(\mathbf{k}, s) = \int \mathrm{d}\mathbf{r}' \, \mathrm{e}^{-i\mathbf{k}\mathbf{r}'} \int_0^\infty \mathrm{d}t' \, \mathrm{e}^{-st'} D(\mathbf{r}', t') \equiv \int \mathrm{d}\mathbf{r}' \, \mathrm{e}^{-i\mathbf{k}\mathbf{r}'} D(\mathbf{r}', s) \,. \tag{2}$$

It is possible to calculate $D(\mathbf{k}, s)$ in terms of equilibrium density fluctuations. To show this we first Laplace transform the equation of continuity, $\dot{\rho} = -\nabla \mathbf{J}$, and calculate the right hand side by means of Eq. (1). If $\tilde{\rho}$ is the Laplace transform of ρ this leads to

$$s\tilde{\rho}(\mathbf{r}, s) - \rho(\mathbf{r}, 0) = \int \mathrm{d}\mathbf{r}' \, D(\mathbf{r}', s) \, \nabla^2 \tilde{\rho}(\mathbf{r} - \mathbf{r}', s) \,. \tag{3}$$

We proceed by Fourier transforming this equation. If $\rho_{\mathbf{k}}$ is the Fourier transform of the density, $\rho_{\mathbf{k}} = \int \mathrm{d}\mathbf{r} \, \rho(\mathbf{r}) \exp(-i\mathbf{k}\mathbf{r})$, one finds

$$s\tilde{\rho}_{\mathbf{k}}(s) - \rho_{\mathbf{k}}(0) = -D(\mathbf{k}, s) \, k^2 \tilde{\rho}_{\mathbf{k}}(s) \,. \tag{4}$$

To arrive at this result it must be assumed that $D(\mathbf{r}', t) \to 0$ as $|\mathbf{r}'| \to \infty$ (otherwise, Eq. (1) does not make sense). Equation (4) implies $\tilde{\rho}_{\mathbf{k}}(s) = \rho_{\mathbf{k}}(0) / \left[s + D(\mathbf{k}, s) \, k^2 \right]$. Multiplying this relation by $\rho_{-\mathbf{k}}(0)$ we find after thermally averaging

$$\frac{\langle \tilde{\rho}_{\mathbf{k}}(s) \, \rho_{-\mathbf{k}}(0) \rangle}{\langle \rho_{\mathbf{k}}(0) \, \rho_{-\mathbf{k}}(0) \rangle} = \frac{1}{s + D(\mathbf{k}, s) \, k^2} \,. \tag{5}$$

We are mainly interested in the $\mathbf{k} \to 0$ limit. The frequency-dependent diffusion constant $D(s)$ is defined by

$$D(s) = \lim_{\mathbf{k} \to 0} D(\mathbf{k}, s) \,. \tag{6}$$

We proceed to derive an equation relating $D(s)$ to the mean-square displacement as function of time, considering from now on only the case of *non-interacting* particles. If d is space dimension and $\Delta\mathbf{r}(t)$ is particle displacement in time t, the expression to be proved is (with an implicit convergence factor $\lim_{\epsilon \to 0} \exp(-\epsilon t) \; \epsilon > 0$)

$$D(s) = \frac{s^2}{2d} \int_0^\infty \langle \Delta\mathbf{r}^2(t) \rangle \, \mathrm{e}^{-st} \, \mathrm{d}t \,. \tag{7}$$

The proof proceeds as follows. If the particle positions are denoted by $\mathbf{r}^{(j)}$ we have $\rho_{\mathbf{k}}(t) = \sum_j \exp[-i\mathbf{k}\mathbf{r}^{(j)}(t)]$. Non-interacting particles move uncorrelated and thus $\langle \rho_{\mathbf{k}}(t) \rho_{-\mathbf{k}}(0) \rangle = \sum_j \langle \exp[-i\mathbf{k} \, \Delta\mathbf{r}^{(j)}(t)] \rangle$. Consequently, it is enough to consider the motion of just one particle and

$$\frac{\langle \rho_{\mathbf{k}}(t) \, \rho_{-\mathbf{k}}(0) \rangle}{\langle \rho_{\mathbf{k}}(0) \rho_{-\mathbf{k}}(0) \rangle} = \langle \mathrm{e}^{-i\mathbf{k} \, \Delta\mathbf{r}(t)} \rangle \,. \tag{8}$$

Being only interested in the limit of small $\mathbf{k}$, the exponential on the right hand side is expanded: $\exp(-i\mathbf{k} \, \Delta\mathbf{r}) = 1 - i\mathbf{k} \, \Delta\mathbf{r} - \frac{1}{2}(\mathbf{k} \, \Delta\mathbf{r})^2 + \ldots$. For the thermal average we have by symmetry $\langle \Delta\mathbf{r} \rangle = 0$ and thus $\langle \exp(-i\mathbf{k} \, \Delta\mathbf{r}) \rangle = 1 - \frac{1}{2} \langle (\mathbf{k} \, \Delta\mathbf{r})^2 \rangle + \ldots$. Spherical symmetry implies that, if Δr_i is the i'th coordinate of $\Delta\mathbf{r}$ etc, $\langle \Delta r_i \, \Delta r_j \rangle = \langle \Delta\mathbf{r}^2 \rangle \, \delta_{ij}/d$. Thus $\langle (\mathbf{k}\Delta\mathbf{r})^2 \rangle = \langle k_i \Delta r_i k_j \Delta r_j \rangle = k^2 \langle \Delta\mathbf{r}^2 \rangle / d$. Substituting this into Eq. (8) with the exponential

expanded to second order, Laplace transforming, and utilizing Eq. (5) leads to

$$\frac{1}{s + D(\mathbf{k}, s)\, k^2} \;=\; \frac{1}{s} - \frac{k^2}{2d} \int_0^\infty \langle \Delta \mathbf{r}^2 (t) \rangle \, \mathrm{e}^{-st} \, \mathrm{d}t \,. \tag{9}$$

For small $\mathbf{k}$ the left hand side is expanded as follows: $s^{-1}\big(1 - s^{-1} D(\mathbf{k}, s)\, k^2 + \ldots\big)$. Comparing this to the right hand side and letting $\mathbf{k} \to 0$ proves Eq. (7).

The case of "ordinary" diffusion is characterized by $\langle \Delta \mathbf{r}^2 (t) \rangle = 2dDt$ where D is the ordinary diffusion constant. Substituting this into Eq. (7) gives $D(s) = D$ at all frequencies. This definition of $D(s)$ is a consistent generalization of the ordinary diffusion constant. Finally, we note that Eq. (7) may be rewritten by performing two partial integrations

$$D(s) \;=\; \frac{1}{d} \int_0^\infty \langle \mathbf{v}(0)\, \mathbf{v}(t) \rangle \, \mathrm{e}^{-st} \, \mathrm{d}t \,. \tag{10}$$

Henceforth we specialize to the case of non-interacting particles hopping on a finite cubic lattice in d dimensions. In order to be able to speak about well-defined displacements we shall *not* impose periodic boundary conditions. This means that $D(0) = 0$. To calculate $D(0)$ in the physically realistic "bulk" limit one should calculate $D(s)$ at a non-zero value of s, let lattice size go to infinity and only thereafter let s go to zero. (The obvious alternative, not used here, is to impose periodic boundary conditions and not consider positions at all but velocities, calculating $D(s)$ by means of Eq. (10).)

Since the particles are assumed to be non-interacting it is only necessary to consider the motion of one particle. A hopping model is defined as follows. Adopting the braket notation of quantum mechanics the lattice sites are denoted by $|j\rangle$. Any state of the system $|P\rangle$ is given by an expression of the form $|P\rangle = \sum_m P_m \, |m\rangle$, where P_m is the probability to find the particle at site m. The state space is a real Hilbert space when equipped with ordinary inner product but, of course, only states with positive probabilities summing to 1 are physical. Time development is described by a master equation which is conveniently written in terms of a "Hamiltonian" as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t} \, |P\rangle \;=\; H \, |P\rangle \,. \tag{11}$$

Here, H is the well-known master equation time-development matrix constructed from the transition probabilities. H has the following properties [6]: All diagonal elements of H are non-positive (giving the decay rate from a state), all off-diagonal elements are non-negative (giving the transition rate between two states). The condition $\sum_j H_{jm} = 0$ ensures probability conservation. Finally, if $P_m^{(0)}$ is the canonical probability to find a particle at site m in thermal equilibrium, for any states $|j\rangle$ and $|m\rangle$ the principle of detailed balance demands that $H_{jm} P_m^{(0)} = H_{mj} P_j^{(0)}$.

Since the solution of Eq. (11) is $|P(t)\rangle = \exp(Ht)|P(0)\rangle$ the mean-square displacement in thermal equilibrium is given by

$$\langle \Delta \mathbf{r}^2(t) \rangle = \sum_{jm} \big(\mathbf{r}_j - \mathbf{r}_m\big)^2 \, \langle j| \, e^{Ht} \, |m\rangle \, P_m^{(0)} \,. \tag{12}$$

As always when dealing with a master equation it is convenient to switch to the "symmetric" representation [6]: First we define the operator T by $T\,|m\rangle = \sqrt{P_m^{(0)}}\,|m\rangle \equiv |\psi_m\rangle$. It is easy to show that the principle of detailed balance implies that the operator $\hat{H} \equiv T^{-1}HT$ is Hermetian (real and symmetric). Furthermore, $\langle j|\exp(Ht)\,|m\rangle = \langle j|\exp\left(T\hat{H}T^{-1}t\right)|m\rangle = \langle j|\,T\exp(\hat{H}t)\,T^{-1}\,|m\rangle = \langle\psi_j|\exp(\hat{H}t)\,|\psi_m\rangle/P_m^{(0)}$. Substituting this into Eq. (12) leads to

$$\langle\Delta\mathbf{r}^2(t)\rangle = \sum_{jm}\left(\mathbf{r}_j-\mathbf{r}_m\right)^2\langle\psi_j|\,\mathrm{e}^{\hat{H}t}\,|\psi_m\rangle\,. \tag{13}$$

From this expression important conclusions are arrived at. First, note that if any state may be reached from any other – the standard ergodic assumption to avoid pathological cases with completely isolated "islands" – $\hat{H}$ has a unique "ground state" with eigenvalue 0 [6]. This state $|0\rangle$ is the symmetric representation of the state of thermal equilibrium. It is straightforward to show that this normalized eigenstate is given by $|0\rangle = \sum_m |\psi_m\rangle$. All other eigenvalues of $\hat{H}$, denoted by $-\gamma_n$, are strictly negative (i.e., $\gamma_n > 0$). If the corresponding eigenstates of $\hat{H}$ are denoted $|n\rangle$, the usual trick of "sandwiching in" the orthonormal set of eigenstates leads to $\langle\psi_j|\exp(\hat{H}t)\,|\psi_m\rangle = P_j^{(0)}P_m^{(0)} + \sum_n \langle\psi_j\mid n\rangle\langle n\mid\psi_m\rangle\exp(-\gamma_n t)$. Substituting this into Eq. (13) leads to

$$\langle\Delta\mathbf{r}^2(t)\rangle = \sum_{jm}\left(\mathbf{r}_j-\mathbf{r}_m\right)^2 P_j^{(0)}P_m^{(0)} - \sum_n \mu_n\,\mathrm{e}^{-\gamma_n t}\,, \tag{14}$$

where $\mu_n \equiv -\sum_{jm}\left(\mathbf{r}_j-\mathbf{r}_m\right)^2\langle\psi_j\mid n\rangle\langle n\mid\psi_m\rangle$. Since $\langle\Delta\mathbf{r}^2(t)\rangle = 0$ for $t=0$ Eq. (14) may be rewritten as

$$\langle\Delta\mathbf{r}^2(t)\rangle \;=\; \sum_n \mu_n(1-\mathrm{e}^{-\gamma_n t})\,. \tag{15}$$

We now proceed to prove that $\mu_n \geq 0$. Suppose that λ_j are real numbers summing to zero. Then $\sum_{jm}\left(\mathbf{r}_j-\mathbf{r}_m\right)^2\lambda_j\lambda_m \leq 0$, because the sum may be rewritten as $\left(\sum_j \lambda_j\mathbf{r}_j^2\right)\left(\sum_m\lambda_m\right) + \left(\sum_m\lambda_m\mathbf{r}_m^2\right)\left(\sum_j\lambda_j\right) - 2\left(\sum_j\lambda_j\mathbf{r}_j\right)^2$ and the first two terms vanish. It now follows that $\mu_n \geq 0$ because for any $|n\rangle$ one has $\sum_j\langle\psi_j|n\rangle = \langle 0|n\rangle = 0$.

We are now able to prove a result for the mathematical structure of the mean-square displacement in hopping models: By repeated differentiations Eq. (15) implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\Delta\mathbf{r}^2(t)\rangle \;\geq 0 \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\;\langle\Delta\mathbf{r}^2(t)\rangle \;\leq 0 \qquad \ldots \text{ etc. } \ldots \tag{16}$$

A related result applies for $D(s)$ when considered as a function of real $s > 0$: Substituting Eq. (15) into Eq. (7) leads to

$$D(s) = \frac{s^2}{2d}\sum_n \mu_n\left(\frac{1}{s}-\frac{1}{s+\gamma_n}\right) = \frac{1}{2d}\sum_n \mu_n\frac{s\gamma_n}{s+\gamma_n}\,. \tag{17}$$

Obviously, $D(s) \geq 0$. Equation (18) is further simplified by writing in the numerator $s = s+\gamma_n-\gamma_n$, leading to

$$D(s) \;=\; D(\infty) \;-\; \frac{1}{2d}\sum_n \mu_n\frac{\gamma_n^2}{s+\gamma_n}\,, \tag{18}$$

where $D(\infty) = \sum_n \mu_n \gamma_n / 2d$. It is straightforward to show that $D(\infty) = \sum_{jm} (\mathbf{r}_j - \mathbf{r}_m)^2 \times \Gamma_{jm} P_m^{(0)} / 2d$ where Γ_{jm} is the transition rate from site $|m\rangle$ to site $|j\rangle$ ($\Gamma_{jm} = -H_{jm}$ for $j \neq m$). Differentiating Eq. (18) with respect to s repeatedly for $s > 0$ leads to

$$\frac{\mathrm{d}}{\mathrm{d}s} D(s) \geq 0 \qquad \frac{\mathrm{d}^2}{\mathrm{d}s^2} D(s) \leq 0 \qquad \ldots \text{ etc. } \ldots \tag{19}$$

Substituting $s = i\omega$ into Eq. (19) immediately gives a result first derived by Kimball and Adams [7]: The real part of the diffusion constant is always an increasing function of ω.

Our final general result again concerns the case $s > 0$, where the approximate power law exponent obeys:

$$0 \leq \frac{\mathrm{d} \ln D}{\mathrm{d} \ln s} \leq 1 \,. \tag{20}$$

The first inequality simply expresses the fact that $D(s)$ is an increasing function of s (Eq. (19)). Now, if $f_1(s)$ and $f_2(s)$ are two positive functions each obeying $\mathrm{d} \ln f / \mathrm{d} \ln s \leq 1$ then this condition is also obeyed for $f_1 + f_2$: We have $s f_1' / f_1 \leq 1$ or $f_1' \leq f_1 / s$ and similarly for f_2. Adding these two inequalities leads to $f_1' + f_2' \leq (f_1 + f_2)/s$ or $\mathrm{d} \ln (f_1 + f_2) / \mathrm{d} \ln s \leq 1$. Next, we note that the function $f(s) = Cs/(s + \gamma)$ obeys $0 \leq \mathrm{d} \ln f / \mathrm{d} \ln s = \gamma / (s + \gamma) \leq 1$ whenever $C \geq 0$ and $\gamma \geq 0$. Equation (20) now follows from Eq. (17).

Universality of the ac Conductivity in the Extreme Disorder Limit If the particles are charged, the frequency-dependent electrical conductivity $\sigma(s)$ is proportional to $D(s)$ (the fluctuation-dissipation theorem). In particular, in hopping models there are relations for $\sigma(s)$ analogous to Eqs. (20) and (22), and the real part of the conductivity, σ' is an increasing function of real frequencies, ω. Hopping models have been discussed for several years in the literature. Some reviews of hopping are listed in Refs. [1–5], and some recent interesting papers dealing with ac hopping are given in Refs. [8–10].

We now specialize to hopping models with only nearest-neighbor jumps, and focus on the simplest and best understood model of this kind, the random barrier model (also called the symmetric hopping model). For this model all sites on the cubic lattice have equal energy. Thus, detailed balance implies for the jump rates $\Gamma_{jm} = \Gamma_{mj}$. Because we want to mimic a disordered solid the jump rates are taken to vary randomly and uncorrelated from link to link. Moreover, we shall assume that the jump rates are given by a free energy barrier E: $\Gamma = \Gamma_0 \exp(-\beta E)$. Here β is the inverse temperature if classical barrier hopping is considered and inverse wave function size if quantum mechanical tunneling is considered. The model is completely defined in terms of the energy barrier probability distribution, $p(E)$.

At large values of β the jump rates vary many orders of magnitude. We term the limit $\beta \to \infty$ the extreme disorder limit. In this limit, although all jump frequencies go to zero and consequently $\sigma \to 0$, one may ask how the conductivity relative to the dc level, $\tilde{\sigma} \equiv \sigma / \sigma(0)$, behaves as a function of frequency. It turns out [11–13] that $\tilde{\sigma}$ as function of a suitably scaled frequency in the extreme disorder limit becomes independent of the barrier distribution $p(E)$. This result, which is referred to as "ac universal-

ity," applies for any smooth $p(E)$ which is non-zero at the percolation threshold. More precisely, ac universality means that for any fixed scaled [Laplace] frequency $\tilde{s}$, as $\beta \to \infty$ the conductivity $\tilde{\sigma}(\tilde{s})$ converges to some value independent of $p(E)$. In computer simulations we find that the larger $\tilde{s}$ is, the larger β must be before there is convergence. Since $\tilde{s} \sim 1$ defines the region below which the conductivity is virtually frequency independent, this means that the convergence to universality is fastest around the onset of ac conduction.

Although no mathematically rigorous proof of ac universality exists, there is convincing evidence for ac universality from several sources [5]. First of all, ac universality is seen clearly in computer simulations, where several quite different $p(E)$'s lead to exactly the same $\tilde{\sigma}(\tilde{s})$ in the extreme disorder limit. Secondly, the effective medium approximation (EMA) implies ac universality in the extreme disorder limit [13]

$$\tilde{\sigma} \ln \tilde{\sigma} = \tilde{s}\,. \tag{21}$$

This equation was first derived by Bryksin as the EMA solution of the hopping model describing electrons tunneling between random positions in space [14]. Finally, it is possible to physically understand the origin of universality; basically universality stems from the fact that *percolation* dominates conduction in the extreme disorder limit [5].

The EMA universality Eq. (23) has the correct qualitative features of the universal ac conductivity: A constant low frequency conductivity and an ac conductivity which at high frequencies follows an approximate power law with an exponent which is below one but converges logarithmically to one as frequency diverges. Quantitatively, however, Eq. (23) is not accurate in three dimensions, and even less accurate in two dimensions. In both cases the onset of ac conduction is less dramatic than predicted by Eq. (23). One may speculate that Eq. (23) for the universal ac conductivity becomes exact above 6 dimensions, because in this regime mean-field theory for percolation becomes exact (when percolation is regarded as a critical phenomenon), and the EMA is a sort of mean-field theory. We have preliminary simulation data for 4 and 5 dimensions indicating that the EMA indeed works better and better as dimension is increased.

Before proceeding to discuss a more accurate analytical approximation to the universal ac conductivity, let us briefly sketch how Eq. (23) is derived because this is relevant for the following. The units used are "rationalized" units where conductivity, diffusion constant and jump rate are all equal in the ordered (homogeneous) case. The EMA equation is a self-consistency equation for $\sigma(s)$ [3]. This equation involves the diagonal element of the Green's function for a random walk on a lattice with uniform jump frequency Γ, $\tilde{G} \equiv \langle \mathbf{r}|\, G_0\, |\mathbf{r}\rangle$, where $G_0 = 1/(s - H_0)$ is the resolvent operator. $\tilde{G}$ is a function of $\Gamma(=\sigma)$, besides of course also a function of s. After some elementary mathematical manipulations the EMA self-consistency equation in the extreme disorder limit reduces [13] to (where C is some constant)

$$\ln \tilde{\sigma} = C\beta s \tilde{G}\,. \tag{22}$$

It is straightforward to show that $s\tilde{G} \propto s/\sigma$ for small s in two and more dimension [13]. When this is substituted into Eq. (24) one arrives at Eq. (23) after a suitable rescaling of s to define the dimensionless quantity $\tilde{s}$

$$\tilde{s} = f(\beta)\frac{s}{\sigma(0)}\,, \tag{23}$$

where $f(\beta) \propto \beta$ for EMA.

As mentioned Eq. (23) does not give an accurate representation of the universal ac conductivity in three dimensions. A much better fit to data is provided by what we term the "diffusion cluster approximation" (DCA), which leads [15] to

$$\ln \tilde{\sigma} = \left(\frac{\tilde{s}}{\tilde{\sigma}}\right)^{d_0/2} . \tag{24}$$

A related equation was derived by Zvyagin in 1980 [16] by reference to the cluster size statistics at percolation (Zvyagin had ω where we have $s = i\omega$). Our simulations in 3D are well fitted by $d_0 = 1.35$. In the original derivation of Eq. (26) d_0 was identified with the dimension of the "diffusion cluster," the subset where the dominant part of the dc conduction takes place. The diffusion cluster is defined as follows. We first recall that at extreme disorder conduction takes place on the percolation cluster [17–19]. More precisely, links with jump rates much smaller than the "percolation jump rate" can be removed without affecting the overall (dc) conductivity of the lattice; for any finite β this leaves us with the "fat" percolation cluster [5] (which becomes the true percolation cluster as $\beta \to \infty$). However, more links may be removed. First of all, those on dead ends of the percolation cluster may be removed. Moreover, on a sufficiently short length scale (to be defined below), if there are two different paths between any two points, because of the extreme disorder, one of them is much more favorable than the other which may be removed. After this "diluting" of the percolation cluster the remainder is by definition the diffusion cluster. Now, our basic assumption is that *in the extreme disorder limit not only dc but also ac conduction takes place mainly on the diffusion cluster*. This assumption runs contrary to all other uses of percolation theory to calculate $\sigma(s)$ that we know of. In principle, the assumption can be checked by computer simulations (work in progress). Please note that, at any finite β, the assumption applies only at not too high frequencies; there are always isolated islands of well-conducting regions which are important at sufficiently high frequencies. Our idea, however, is that the [scaled] frequency below which the conjecture applies diverges as $\beta \to \infty$.

What is the structure of the diffusion cluster? Let us first recall the "nodes-links-blobs" model of the backbone of the percolation cluster (the part which carries dc current in the case of only two jump rates, 0 or 1). According to this model [20] the backbone comprises links (i.e., quasi one-dimensional strings) and nodes at the intersection of links. The "blobs" refer to the fact that for $0-1$ percolation there are occasional strongly bonded regions along any link. However, as argued above, blobs are unimportant in the extreme disorder limit of a continuum distribution of jump rates because, of any two different paths between two points one will strongly dominate.

The picture is now the following: At any finite β, the time scales on which a particle moves more than the node-node distance correspond to frequencies where the conductivity is frequency independent. On a smaller distance scale the diffusion cluster is fractal. In our original derivation of Eq. (26) we used the EMA equation Eq. (24) *for hopping on the percolation cluster* and identified d_0 with the dimension of the diffusion cluster, having in mind the fractal dimension $\bar{d}$. More correctly, d_0 should be identified with the so-called harmonic or fracton dimension $\bar{\bar{d}}$ which for [homogeneous] random walks on the diffusion cluster after time t gives the probability $P(t)$ to be at the same place as at $t = 0$: $P(t) \propto t^{-\bar{\bar{d}}/2}$ and $\tilde{G}_0$ is basically the Laplace transform of $P(t)$

[$s\tilde{\boldsymbol{G}} \propto (s/\sigma)^{\bar{\bar{d}}/2}$]. The connection between $\bar{\bar{d}}$ and $\bar{d}$ is [20, 21]

$$\bar{\bar{d}} = \frac{2\bar{d}}{2+\delta}\,, \tag{25}$$

where the exponent δ is given by $\langle \Delta \mathbf{r}^2(t) \rangle \propto t^{2/(2+\delta)}$ (it important to remember that these relations refer to random walks on the diffusion cluster with a *uniform jump rate* – only the geometry of the diffusion cluster matters). Is it possible that $\bar{d} = \bar{\bar{d}}$, corresponding to $\delta = 0$? In our opinion the answer is yes, because $\delta = 0$ for any fractal without dead ends and without loops, e.g., a selfsimilar curve like the Koch curve [20, 22]. If the links of the above "nodes-links model" for the diffusion cluster are of this kind one has $\bar{d} = \bar{\bar{d}}$. Indeed, we argued above that the diffusion cluster has no dead ends and no loops. Another interpretation of our finding that $d_0 = 1.35$ is the following: Since $d_o = \bar{\bar{d}}$ the Alexander-Orbach conjecture immediately comes to mind. According to this $\bar{\bar{d}} = 4/3$ for the percolation cluster (including loops and dead ends). Numerically one finds that $\bar{\bar{d}} = 1.32$ not quite 4/3, and not quite 1.35. Nevertheless, our simulations may be taken to indicate that the diffusion cluster is indeed identical to the entire percolation cluster. This hypothesis may be investigated by simulations (work in progress).

We finally briefly discuss the temperature scaling of the dimensionless frequency in the DCA. As mentioned, for random walks on the diffusion cluster $s\tilde{\boldsymbol{G}} \propto (s/\sigma)^{d_0/2}$ where $d_0 = \bar{\bar{d}}$. When this is substituted into Eq. (24) we find that the function $f(\beta)$ in Eq. (25) is given by $f(\beta) \propto \beta^{2/d_0}$. For $d_0 = 1.35$ we find $f \propto \beta^{1.48}$. In our simulations we find a similar power law for $f(\beta)$, albeit with exponent 1.37 [23]. Roling finds an exponent equal to 1.3 [10].

Some Open Problems Many questions regarding ac universality in the extreme disorder limit have not yet been answered:

For the random barrier model: How does the universal ac conductivity depend on dimensionality? In particular: Is Eq. (23) exact in six dimensions and above?

For the more general asymmetric hopping model, i.e., with lattice sites of differing energies: Does this model also have ac universality in the extreme disorder limit? If yes: Does the universal ac conductivity depend on the choice of transition rates? How does it depend on dimensionality?

More realistically one should consider the hopping of particles with only room for one at each site (i.e., Fermions). For Fermi hopping: Is there ac universality in the extreme disorder limit? If yes: Is this the same as the ac universality for the random barrier model (which is the mean-field [Hartree] limit of the Fermi model [18, 24])?

In our opinion much remains to be done in this challenging field of research. Compared to the 1970's and 1980's we now have the possibility of extensive computer simulations at hand. This is likely to bring further progress in the field.

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